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Geometry

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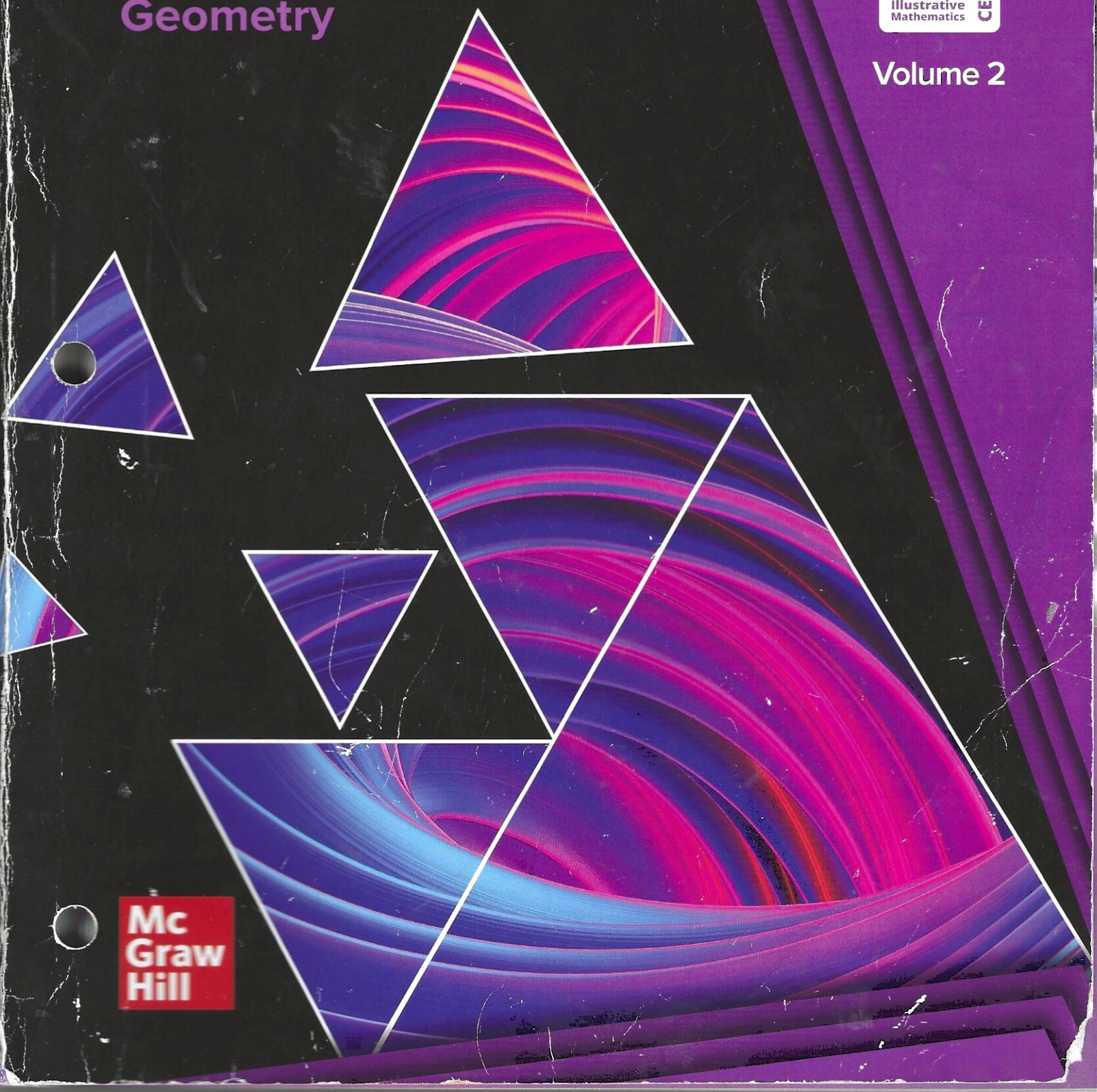
Volume 2



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Part 3 of 4–Unit 7

Volume 2

McGraw Hill

Illustrative Mathematics®

Geometry

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McGraw Hill
STEM Learning Solutions Center
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Student Edition, Volume 2

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Illustrative Mathematics is designed to provide opportunities for you to learn math by doing math. By engaging in problem-solving activities, you will make and test your own conclusions and try out different strategies as you learn math.

Let's get started!

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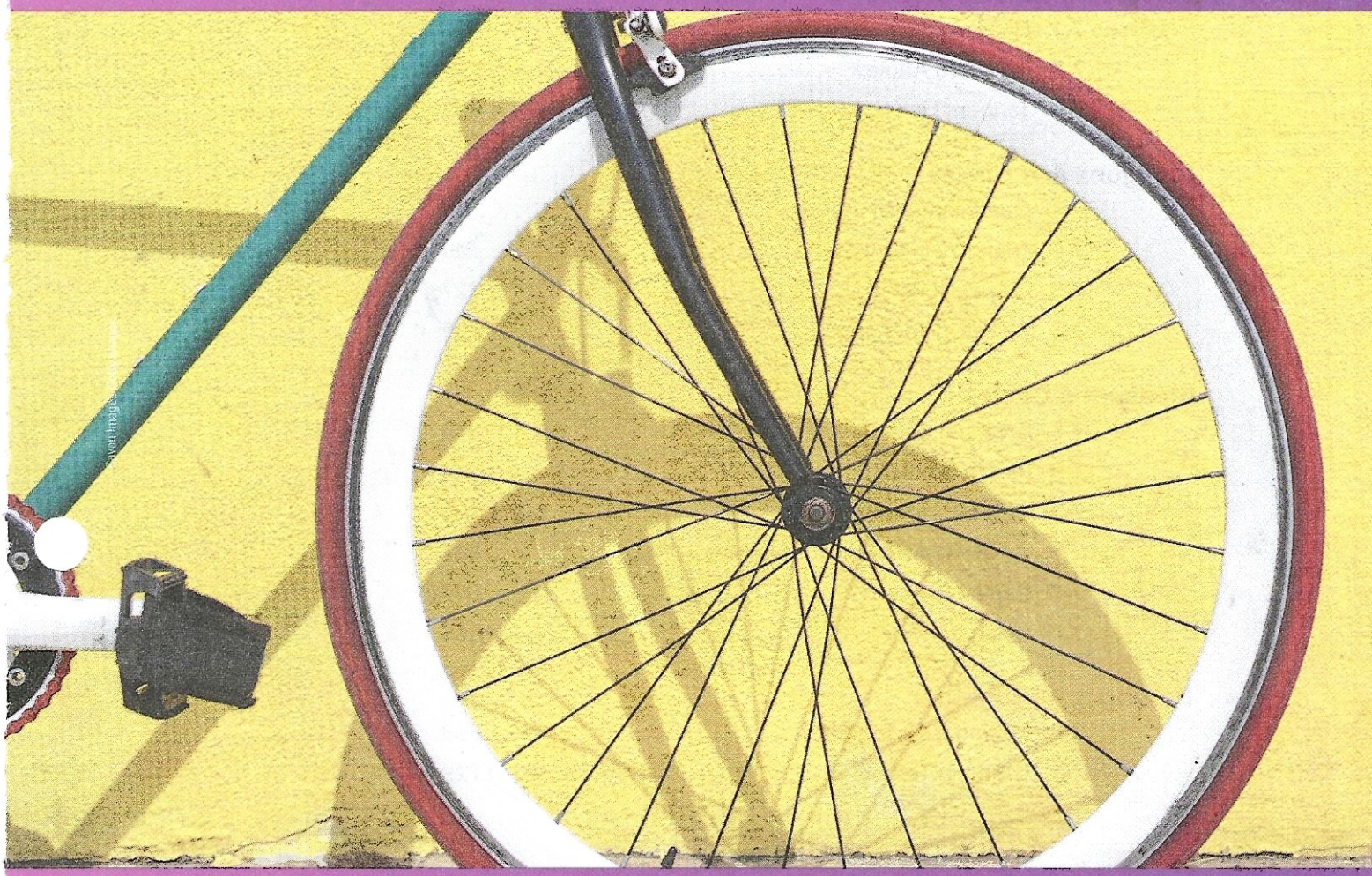
Unit 7

Circles



The gears and tires of a bicycle can be modeled by circles. The diameter of a circle is related to its circumference. You'll learn more about circles in this unit.

Topics

- Lines, Angles, and Circles
- Polygons and Circles
- Measuring Circles
- Putting It All Together

Unit 7

Circles

Lines, Angles, and Circles

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Lesson 7-1

Lines, Angles, and Curves

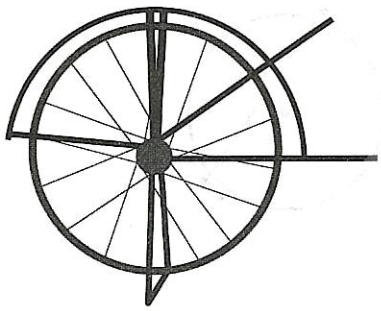
NAME _____ DATE _____ PERIOD _____

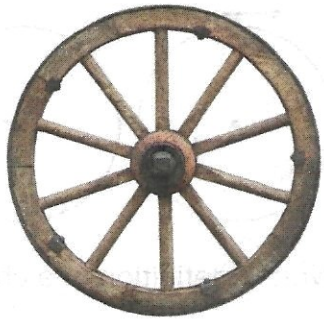
Learning Goal Let's define some line segments and angles related to circles.

Warm Up

1.1 Notice and Wonder: Lines and Angles

What do you notice? What do you wonder?





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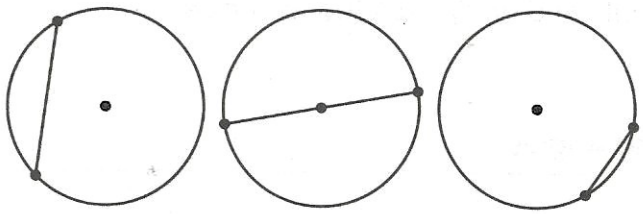
Activity

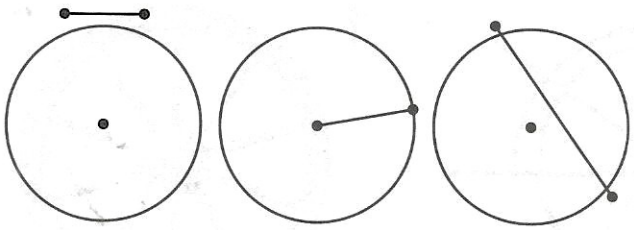
1.2 The Defining Moment

1. The images show some line segments that are **chords** and some segments that are not chords.

chords



not chords

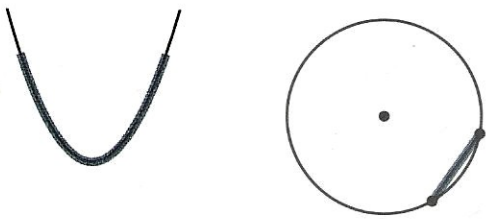


Write a definition of a chord.

2. The images show some highlighted objects that are **arcs**, and some highlighted objects that are not arcs.

arcs

not arcs

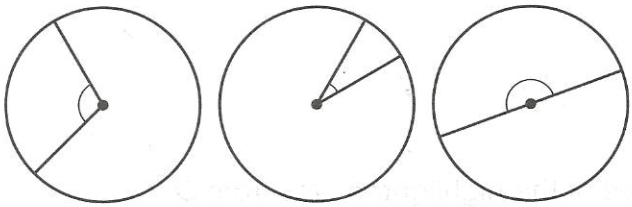


Write a definition of an arc.

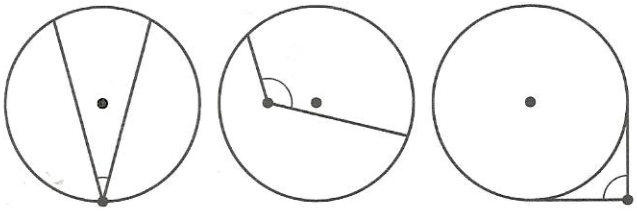
NAME _____ DATE _____ PERIOD _____

3. The images show some angles that are **central angles**, and some that are not.

central angles



not central angles



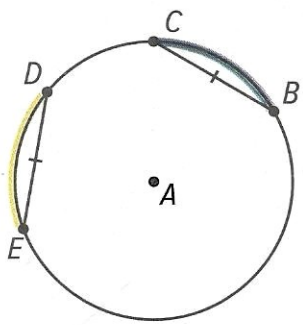
Write a definition of a central angle.

Activity

1.3 Arcs, Chords, and Central Angles

The image shows a circle with 2 congruent chords.

1. Draw the central angles associated with the highlighted arcs from D to E and B to C.



2. How do the measures of the 2 central angles appear to compare? Prove that this observation is true.

3. What does this tell you about the measures of the highlighted arcs from D to E and B to C? Explain your reasoning.

Are you ready for more?

Prove that the perpendicular bisector of a chord goes through the center of a circle.

NAME _____ DATE _____ PERIOD _____

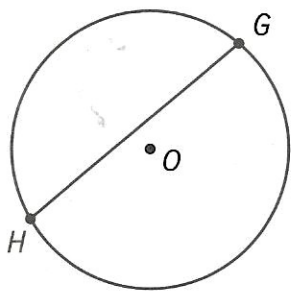
Summary

Lines, Angles, and Curves

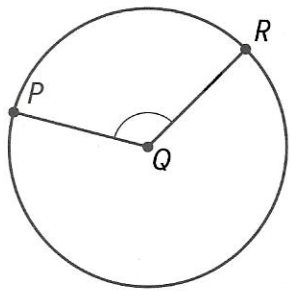
Diameters and radii are 2 types of line segments that appear in circles. Here are some additional geometric objects associated with circles.

A **chord** is a line segment whose endpoints are on the circle. A **central angle** in a circle is an angle whose vertex is at the center of the circle. An **arc** is the portion of a circle between 2 points on the circle. The measure of an arc is defined as the measure of the central angle formed by the radii drawn to the endpoints of the arc. For example, in the image, the highlighted arc between points *D* and *E* measures 45 degrees because the central angle *DAE* measures 45 degrees.

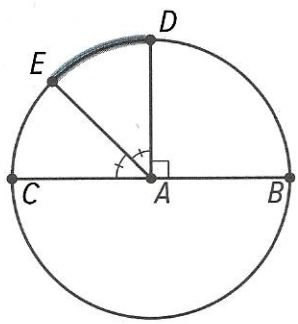
chord *GH*



central angle *PQR*



arc *DE*



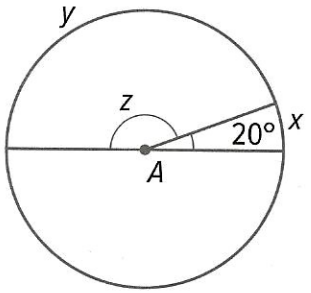
Glossary

arc
central angle
chord

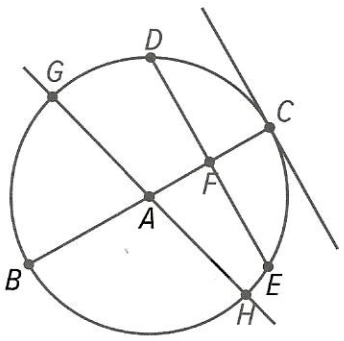
Practice

Lines, Angles, and Curves

1. Find the values of x, y, and z.



2. Give an example from the image of each kind of segment.



a. a diameter

b. a chord that is not a diameter

c. a radius

3. Identify whether each statement must be true, could possibly be true, or definitely can't be true.

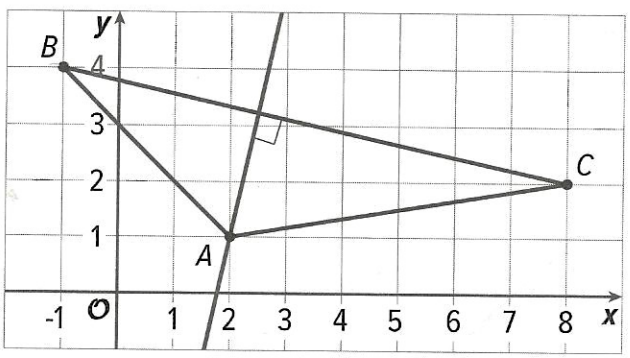
a. A diameter is a chord.

b. A radius is a chord.

c. A chord is a diameter.

d. A central angle measures 90°.

4. Write an equation of the altitude from vertex A. (Lesson 6-17)



NAME _____ DATE _____ PERIOD _____

5. Triangle *ABC* has vertices at (5, 0), (1, 6), and (9, 3). What is the point of intersection of the triangle's medians? (Lesson 6-16)

(A.) The medians do not intersect in a single point.

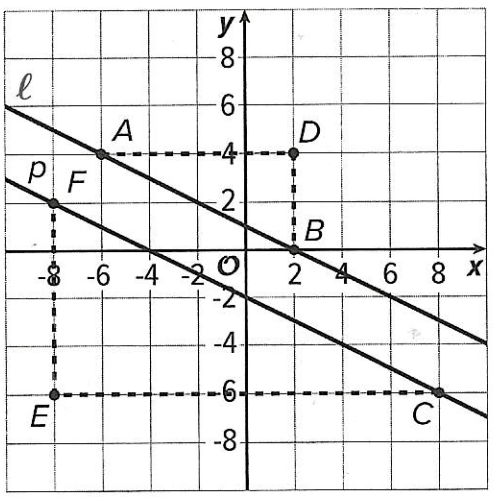
(B.) (3, 3)

(C.) (5, 3)

(D.) (3, 4.5)

6. Consider the parallelogram with vertices at (0, 0), (8, 0), (4, 6), and (12, 6). Where do the diagonals of this parallelogram intersect? (Lesson 6-15)

7. Lines ℓ and p are parallel. Select **all** true statements. (Lesson 6-10)



(A.) Triangle *ADB* is congruent to triangle *CEF*.

(B.) The slope of line ℓ is equal to the slope of line p.

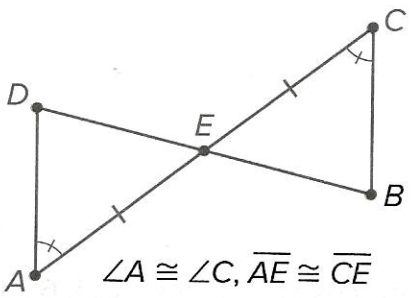
(C.) Triangle *ADB* is similar to triangle *CEF*.

(D.) $\sin(A) = \sin(C)$

(E.) $\cos(B) = \sin(C)$

8. Mai wrote a proof that triangle *AED* is congruent to triangle *CEB*. Mai's proof is incomplete. How can Mai fix her proof?

We know side *AE* is congruent to side *CE* and angle *A* is congruent to angle *C*. By the Angle-Side-Angle Triangle Congruence Theorem, triangle *AED* is congruent to triangle *CEB*. **(Lesson 2-7)**



Lesson 7-2

Inscribed Angles

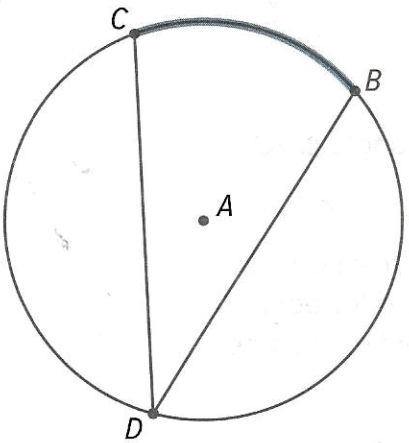
NAME _____ DATE _____ PERIOD _____

Learning Goal Let's analyze angles made from chords.

Warm Up

2.1 Notice and Wonder: A New Angle

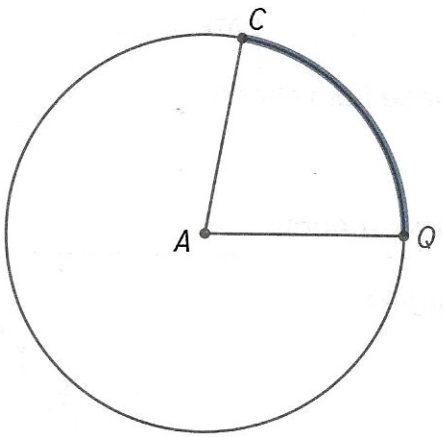
What do you notice? What do you wonder?



Activity

2.2 A Central Relationship

Here is a circle with central angle QAC.



1. Use a protractor to find the approximate degree measure of angle QAC.

2. Mark a point B on the circle that is *not* on the highlighted arc from C to Q. Each member of your group should choose a different location for point B. Draw chords BC and BQ. Use a protractor to find the approximate degree measure of angle QBC.

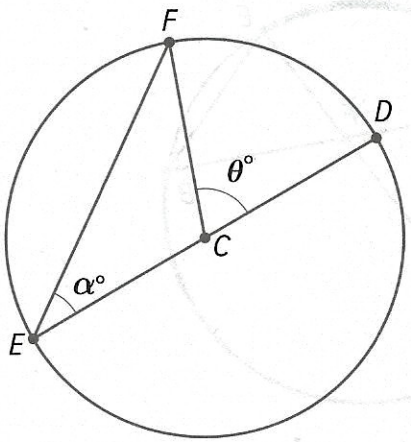
3. Share your results with your group. What do you notice about your answers?

4. Make a conjecture about the relationship between an **inscribed angle** and the central angle that defines the same arc.

NAME _____ DATE _____ PERIOD _____

Are you ready for more?

Here is a special case of an inscribed angle where one of the chords that defines the inscribed angle goes through the center. The central angle *DCF* measures θ degrees, and the inscribed angle *DEF* measures α degrees. Prove that $\alpha = \frac{1}{2}\theta$.

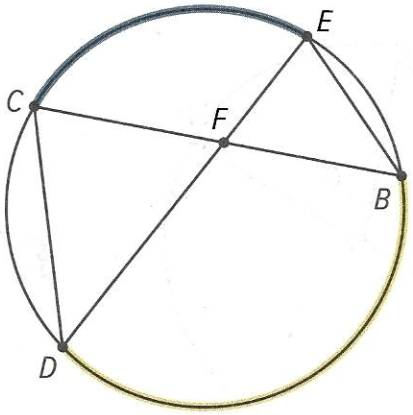


Activity

2.3 Similarity Returns

The image shows a circle with chords *CD*, *CB*, *ED*, and *EB*. The highlighted arc from point *C* to point *E* measures 100 degrees. The highlighted arc from point *D* to point *B* measures 140 degrees.

Prove that triangles *CFD* and *EFB* are similar.

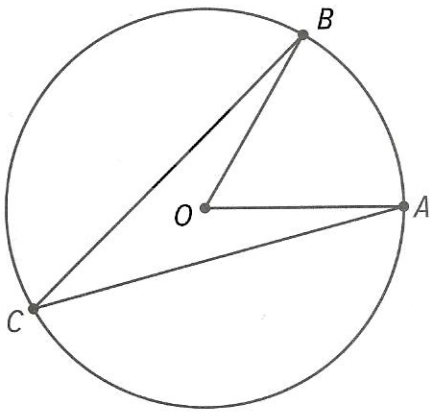


NAME _____ DATE _____ PERIOD _____

Summary

Inscribed Angles

We have discussed central angles such as angle *AOB*. Another kind of angle in a circle is an **inscribed angle**, or an angle formed by 2 chords that share an endpoint. In the image, angle *ACB* is an inscribed angle.



It looks as though the inscribed angle is smaller than the central angle that defines the same arc. In fact, the measure of an inscribed angle is always exactly half the measure of the associated central angle. For example, if the central angle *AOB* measures 50 degrees, the inscribed angle *ACB* must measure 25 degrees, even if we move point *C* along the circumference (without going past *A* or *B*). This also means that all inscribed angles that define the same arc are congruent.

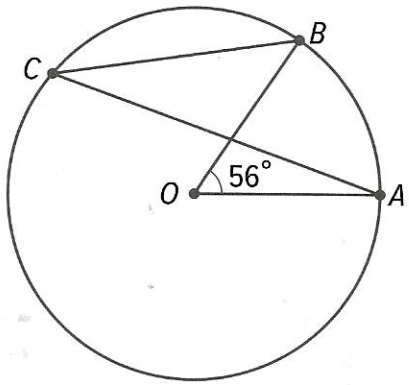
Glossary

inscribed angle

Practice

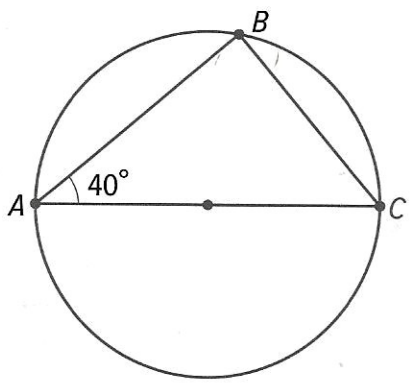
Inscribed Angles

1. The measure of angle *AOB* is 56 degrees. What is the measure of angle *ACB*?



2. Explain the difference between central and inscribed angles.

3. What is the measure of the arc from *A* to *B* that does not pass through *C*?



(A.) 160 degrees

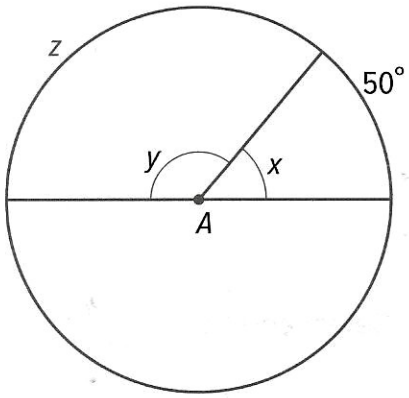
(B.) 140 degrees

(C.) 100 degrees

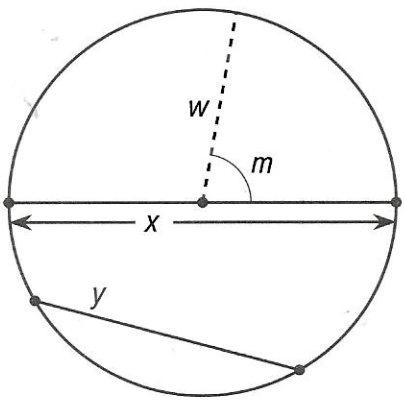
(D.) 90 degrees

NAME _____ DATE _____ PERIOD _____

4. Find the values of x, y, and z. **(Lesson 7-1)**



5. Match the vocabulary term with the label. **(Lesson 7-1)**



A. chord that is not a diameter	1. m
B. diameter	2. w
C. radius	3. x
D. central angle	4. y

6. Triangle ABC has vertices at (-4, 0), (-2, 12), and (12, 0). What is the point of intersection of its medians? (Lesson 6-16)

(A.) (4, 0)

(B.) (5, 6)

(C.) (2, 4)

(D.) (4, 2)

7. The rule $(x, y) \rightarrow (y, -x)$ takes a line to a perpendicular line. Select another rule that takes a line to a perpendicular line. (Lesson 6-11)

(A.) $(x, y) \rightarrow (-y, -x)$

(B.) $(x, y) \rightarrow (2y, 2x)$

(C.) $(x, y) \rightarrow (-4y, 4x)$

(D.) $(x, y) \rightarrow (0.25y, -4)$

Lesson 7-3

Tangent Lines

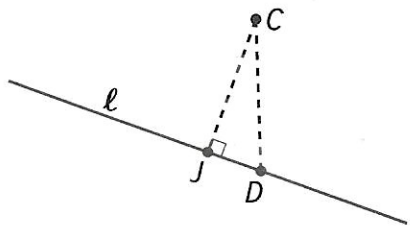
NAME _____ DATE _____ PERIOD _____

Learning Goal Let's explore lines that intersect a circle in exactly 1 point.

Warm Up

3.1 Swim to Shore

Line ℓ represents a straight part of the shoreline at a beach. Suppose you are in the ocean at point C and you want to get to the shore as fast as possible. Assume there is no current. Segments CJ and CD represent 2 possible paths.



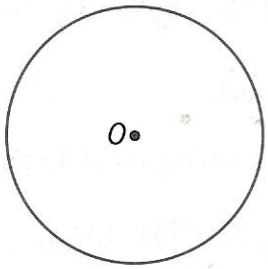
Diego says, "No matter where we put point D, the Pythagorean Theorem tells us that segment CJ is shorter than segment CD. So, segment CJ represents the shortest path to shore."

Do you agree with Diego? Explain your reasoning.

Activity

3.2 A Particular Perpendicular

1. Draw a radius in the circle. Mark the point where the radius intersects the circle and label it A.
2. Construct a line perpendicular to the radius that goes through point A. Label this line n.
3. Line n intersects the circle in exactly 1 point, A. Why is it impossible for line n to intersect the circle in more than 1 point?

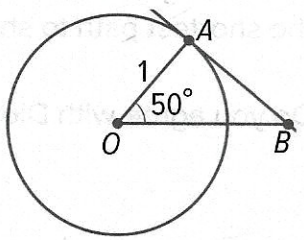


4. What kind of line, then, is n?

Are you ready for more?

Here is a circle centered at O with radius 1 unit. Line AB is tangent to the circle.

1. Calculate the length of segment AB.
2. How does the length of segment AB relate to the tangent of 50 degrees? Why is this true?



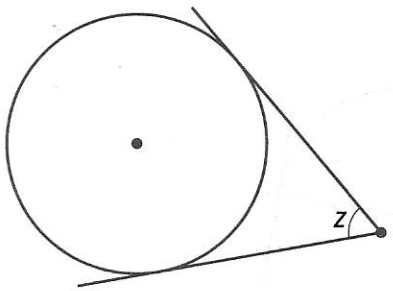
3. In right triangle trigonometry, "tangent" is defined in terms of side ratios. Write another definition of tangent (in the sense of a numerical value, not a line) in terms of a circle with radius 1 unit.

NAME _____ DATE _____ PERIOD _____

Activity

3.3 Another Angle

The image shows an angle whose rays are **tangent** to a circle.

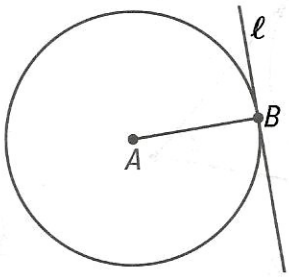


1. Mark the approximate points of tangency.
2. Draw the 2 radii that intersect these points of tangency. Label the measure of the central angle that is formed w.
3. What is the value of $w + z$? Explain or show your reasoning.

Summary

Tangent Lines

A line is said to be **tangent** to a circle if it intersects the circle in exactly 1 point. Suppose line ℓ is tangent to a circle centered at A. Draw a radius from the center of the circle to the point of tangency, or the point where line ℓ intersects the circle. Call this point B. It looks like radius AB is perpendicular to line ℓ. Can we prove it?



Every other point on the tangent line is outside the circle, so they must all be further away from the center than the point where the tangent intersects. This means the point B where the tangent line intersects the circle is the closest point on the line to the center point A. The radius AB must be perpendicular to the tangent line because the shortest distance from a point to a line is always along a perpendicular path.

Glossary

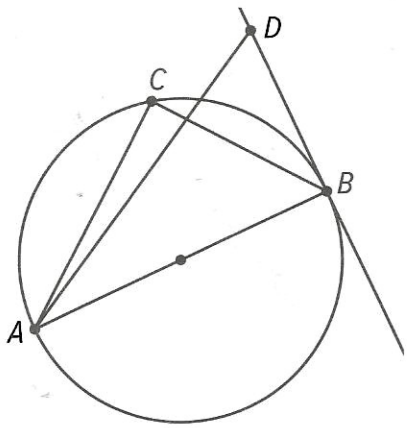
tangent (line)

NAME _____ DATE _____ PERIOD _____

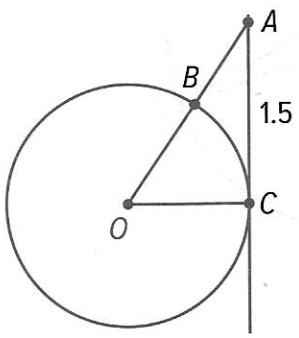
Practice

Tangent Lines

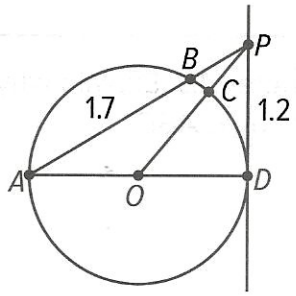
1. Line BD is tangent to a circle with diameter AB. Explain why the measure of angle BCA must equal the measure of angle ABD.



2. Line AC is perpendicular to the circle centered at O with radius 1 unit. The length of AC is 1.5 units. Find the length of segment AB.



3. *Technology required.* Line PD is tangent to a circle of radius 1 inch centered at O. The length of PD is 1.2 inches. The length of AB is 1.7 inches. Which point on the circle is closest to point P?



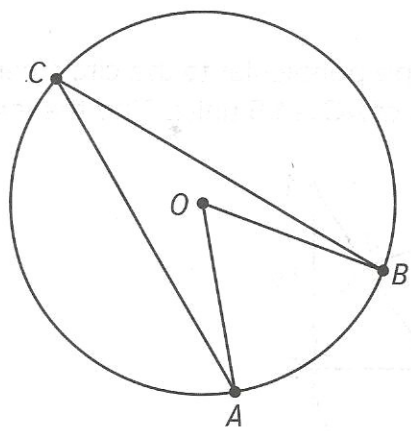
(A.) point A

(B.) point B

(C.) point C

(D.) point D

4. The arc from A to B not passing through C measures 50 degrees. Select **all** the true statements. (Lesson 7-2)



(A.) Angle BCA measures 50 degrees.

(B.) Angle BCA measures 25 degrees.

(C.) Angle BOA measures 50 degrees.

(D.) The arc from B to C not passing through A measures 180 degrees.

(E.) Angles CBO and CAO are congruent.

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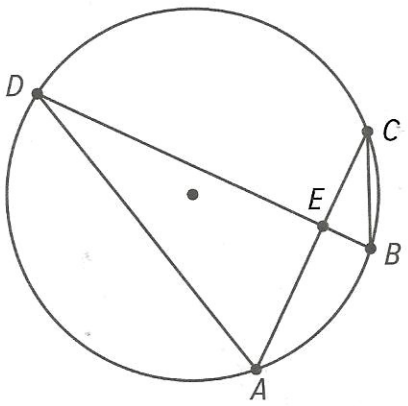
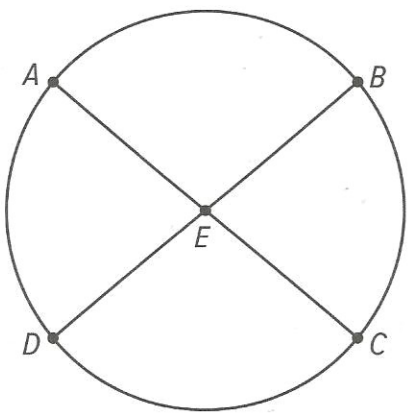
5. Chords *AC* and *DB* intersect at point *E*. List 3 pairs of angles that *must* be congruent. **(Lesson 7-2)**



6. The image shows a circle with diameters *AC* and *BD*. Prove that chords *BC* and *AD* (not drawn) are congruent. **(Lesson 7-1)**



7. The line represented by $y + 3 = -3(x + 6)$ is transformed by the rule $(x, y) \rightarrow (-x, -y)$. What is the slope of the image? **(Lesson 6-12)**

(A.) 3

(B.) $\frac{1}{3}$

(C.) $-\frac{1}{3}$

(D.) -3

Lesson 7-4

Quadrilaterals in Circles

NAME _____ DATE _____ PERIOD _____

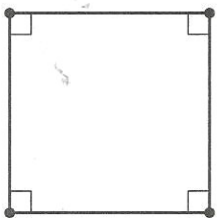
Learning Goal Let's investigate quadrilaterals that fit in a circle.

Warm Up

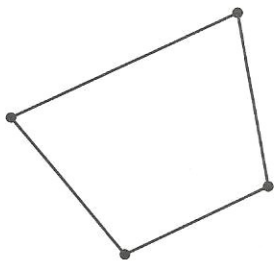
4.1 Connecting the Dots

For each quadrilateral, use a compass to see if you can draw a circle that passes through all 4 of the quadrilateral's vertices.

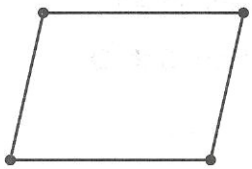
A



B



C

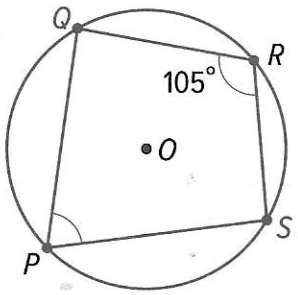


Activity

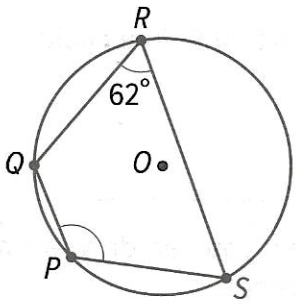
4.2 Inscribed Angles and Circumscribed Circles

1. The images show 3 quadrilaterals with **circumscribed** circles.

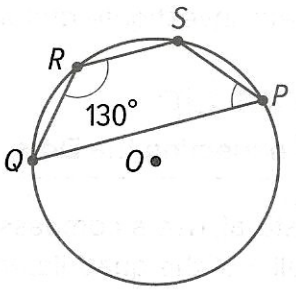
A



B



C



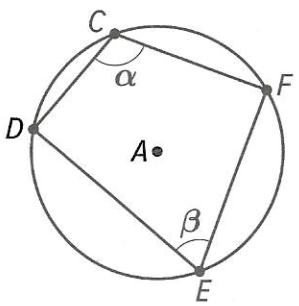
For each one, highlight the arc from S to Q passing through P. Then, find the measures of:

a. the arc you highlighted

b. the other arc from S to Q

c. angle SPQ

2. Here is another quadrilateral with a circumscribed circle. What is the value of $\alpha + \beta$? Explain or show your reasoning.



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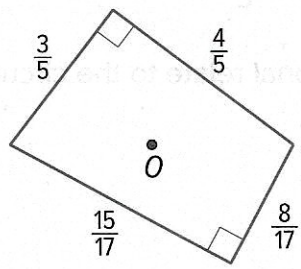
Are you ready for more?

Brahmagupta's formula states that for a quadrilateral whose vertices all lie on the same circle, the area of the quadrilateral is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ where a, b, c, and d are the lengths of the quadrilateral's sides and s is half its perimeter.

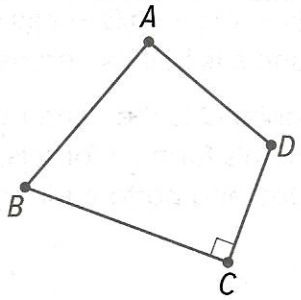
In the cyclic quadrilateral in the image, point O is the center of the quadrilateral's circumscribed circle. Validate Brahmagupta's formula for this particular quadrilateral by first finding the sum of the areas of the top and bottom triangles. Then, calculate the area again using Brahmagupta's formula.



Activity

4.3 Construction Ahead

Quadrilateral *ABCD* is a **cyclic quadrilateral**.



1. Draw diagonal *BD*. How will this diagonal relate to the circumscribed circle? Explain your reasoning.

2. Construct the center of the circumscribed circle for quadrilateral *ABCD*. Label this point *O*. Explain why your method worked.

3. Construct the circumscribed circle for quadrilateral *ABCD*.

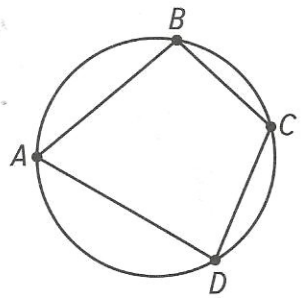
4. Could we follow this procedure to construct a circumscribed circle for *any* cyclic quadrilaterals? Explain your reasoning.

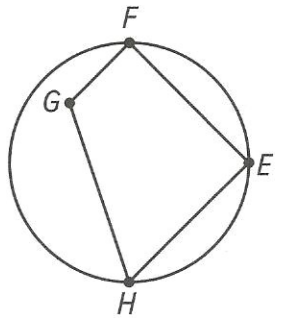
NAME _____ DATE _____ PERIOD _____

Summary

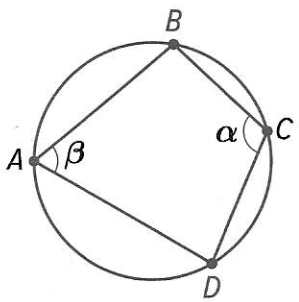
Quadrilaterals in Circles

A circle is said to be **circumscribed** about a polygon if all the vertices of the polygon lie on the circle. If it is possible to draw a circumscribed circle for a quadrilateral, the figure is called a **cyclic quadrilateral**. Not all quadrilaterals have this property.





We can prove that the opposite angles of a cyclic quadrilateral are supplementary. Consider opposite angles *BCD* and *BAD*, labeled α and β.



Angle *BAD* is inscribed in the arc from *B* to *D* through *C*. Angle *BCD* is inscribed in the arc from *B* to *D* through *A*. Together, the 2 arcs trace out the entire circumference of the circle, so their measures add to 360 degrees. By the Inscribed Angle Theorem, the sum of α and β must be half of 360 degrees, or 180 degrees. So angles *BAD* and *BCD* are supplementary. The same argument can be applied to the other pair of opposite angles.

Glossary

circumscribed
cyclic quadrilateral

Practice

Quadrilaterals in Circles

1. A quadrilateral *ABCD* has the given angle measures. Select **all** measurements which could come from a cyclic quadrilateral.

   (A.) angle *A* is 90°, angle *B* is 90°, angle *C* is 90°, and angle *D* is 90°

   (B.) angle *A* is 80°, angle *B* is 80°, angle *C* is 100°, and angle *D* is 100°

   (C.) angle *A* is 70°, angle *B* is 110°, angle *C* is 70°, and angle *D* is 110°

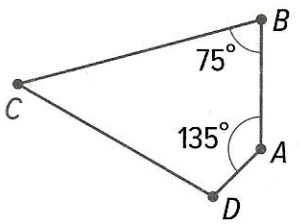
   (D.) angle *A* is 60°, angle *B* is 50°, angle *C* is 120°, and angle *D* is 130°

   (E.) angle *A* is 50°, angle *B* is 40°, angle *C* is 120°, and angle *D* is 150°

2. Quadrilateral *ABCD* is cyclic with given angle measures.

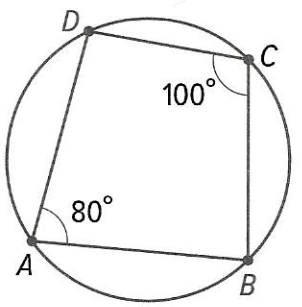
   a. What is the measure of angle *C*?

   b. What is the measure of angle *D*?



3. Lin is looking at cyclic quadrilateral *ABCD*. She says, "I'm not convinced that opposite angles of cyclic quadrilaterals always add up to 180 degrees. For example, in this diagram, suppose we moved point *A* to a different spot on the circle. Angle *BCD* would still measure 100 degrees, but now angle *BAD* would have a different measure, and they wouldn't add up to 180."

   Do you agree with Lin? Explain or show your reasoning.



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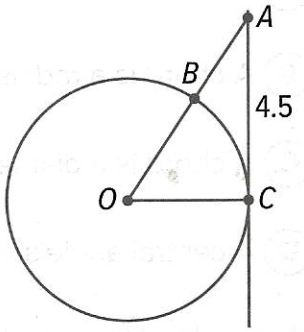
4. Line AC is tangent to the circle centered at O with radius 3 units. The length of segment AC is 4.5 units. Find the length of segment AB. (Lesson 7-3)



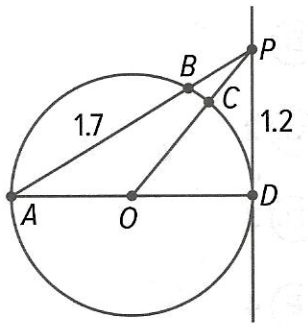
(A.) $3 + \sqrt{29.25}$ units

(B.) $\sqrt{29.25}$ units

(C.) $-3 + \sqrt{29.25}$ units

(D.) 26.25 units

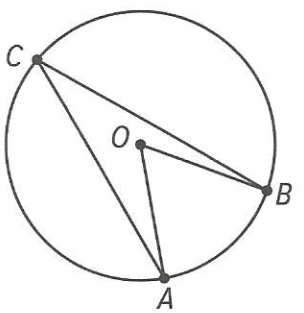
5. *Technology required.* Line PD is tangent to a circle of radius 1 inch centered at O. The length of segment PD is 1.2 inches. The length of segment AB is 1.7 inches. Han is trying to figure out if C or B is closer to P. He uses the Pythagorean Theorem to find the length of OP. Then he subtracts 1 from the length of OP to determine how far C is from point P. (Lesson 7-3)



a. How far is B from point P?

b. Which point is closest to P? Explain your reasoning.

6. In the diagram, the measure of angle ACB is 25 degrees. What is the measure of angle AOB? (Lesson 7-2)



7. Which statement **must** be true? (Lesson 7-1)

(A.) A diameter is a chord.

(B.) A chord is a radius.

(C.) A chord is a diameter.

(D.) A central angle's vertex is on the circle.

8. A circle and line are drawn. How many intersection points are possible? Select **all** possible answers. (Lesson 6-13)

(A.) 0

(B.) 1

(C.) 2

(D.) 3

(E.) 4

Lesson 7-5

Triangles in Circles

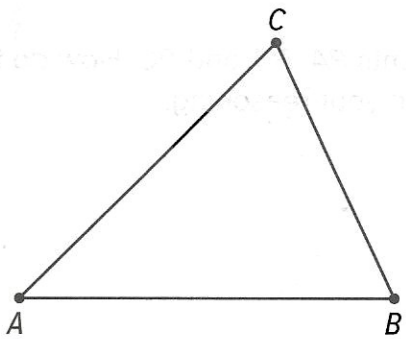
NAME _____ DATE _____ PERIOD _____

Learning Goal Let's see how perpendicular bisectors relate to circumscribed circles.

Warm Up

5.1 One Perpendicular Bisector

The image shows a triangle.



1. Construct the perpendicular bisector of segment AB.

2. Imagine a point D placed anywhere on the perpendicular bisector you constructed. How would the distance from D to A compare to the distance from D to B? Explain your reasoning.

Activity

5.2 Three Perpendicular Bisectors

1. Construct the perpendicular bisector of segment *BC* from the earlier activity. Label the point where the 2 perpendicular bisectors intersect as *P*.

2. Use a colored pencil to draw segments *PA*, *PB*, and *PC*. How do the lengths of these segments compare? Explain your reasoning.

3. Imagine the perpendicular bisector of segment *AC*. Will it pass through point *P*? Explain your reasoning.

4. Construct the perpendicular bisector of segment *AC*.

5. Construct a circle centered at *P* with radius *PA*.

6. Why does the circle also pass through points *B* and *C*?

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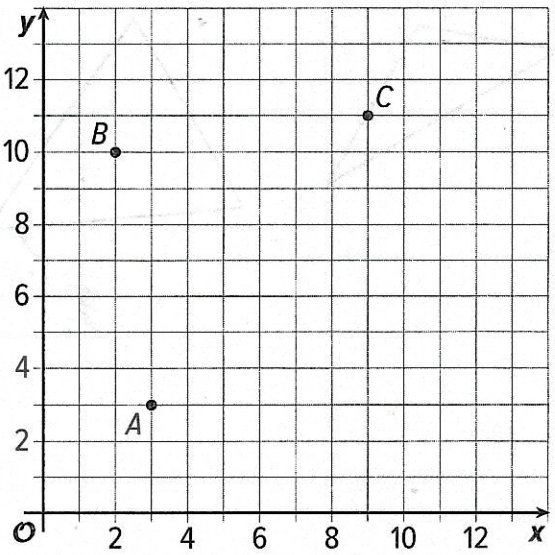
Are you ready for more?

Points A, B, and C are graphed. Find the coordinates of the circumcenter and the radius of the circumscribed circle for triangle ABC.

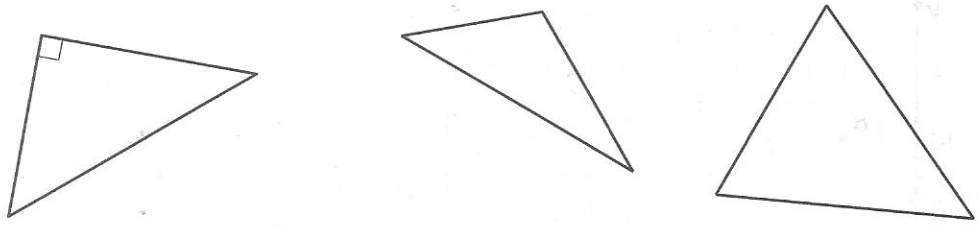


Activity

5.3 Wandering Centers

Each student in your group should choose 1 triangle. It's okay for 2 students to choose the same triangle as long as all 3 are chosen by at least 1 student.

1. Construct the circumscribed circle of your triangle.



2. After you finish, compare your results. What do you notice about the location of the **circumcenter** in each triangle?

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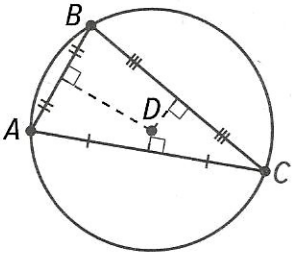
Summary

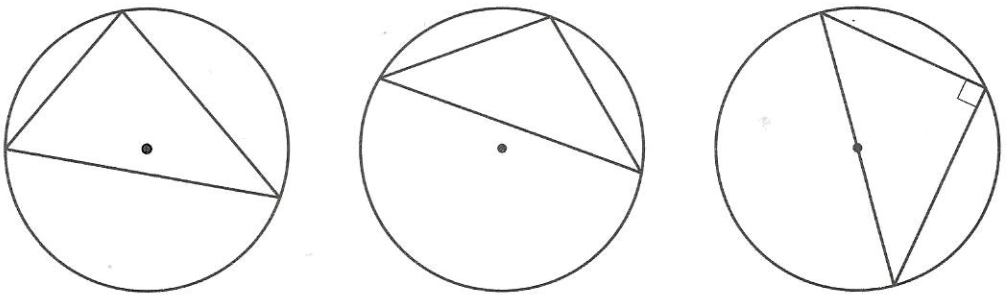
Triangles in Circles

We saw that some quadrilaterals have circumscribed circles. Is the same true for triangles? In fact, *all* triangles have circumscribed circles. The key fact is that all points on the perpendicular bisector of a segment are equidistant from the endpoints of the segment.



Suppose we have triangle ABC and we construct the perpendicular bisectors of all 3 sides. These perpendicular bisectors will all meet at a single point called the **circumcenter** of the triangle (label it D). This point is on the perpendicular bisector of AB, so it's equidistant from A and B. It's also on the perpendicular bisector of BC, so it's equidistant from B and C. So, it is actually the same distance from A, B, and C. We can draw a circle centered at D with radius AD. The circle will pass through B and C too because the distances BD and CD are the same as the radius of the circle.

In this case, the circumcenter happened to fall inside triangle ABC, but that does not need to happen. The images show cases where the circumcenter is inside a triangle, outside a triangle, and on one of the sides of a triangle.



Glossary

circumcenter

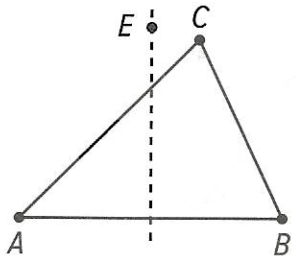
Practice

Triangles in Circles

1. Noah says, "I constructed 2 perpendicular bisectors of triangle *ABC*. That means the point where they intersect is the circumcenter!" Andre responds, "No, we still need to check the third perpendicular bisector to make sure it intersects at the same point."

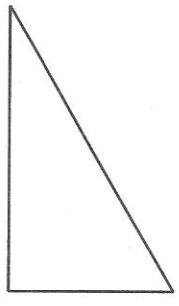
   Do you agree with either of them? Explain or show your reasoning.

2. The dotted line is the perpendicular bisector of side *AB*. The distance between points *E* and *A* is 7 units. What is the distance between points *E* and *B*? Explain or show your reasoning.

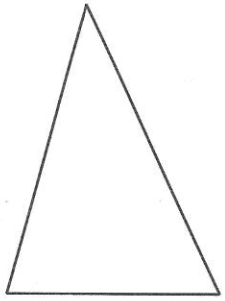


3. Construct the circumcenter of each triangle. Then, based on the locations of the circumcenters, classify each triangle as acute, right, or obtuse.

triangle A



triangle B



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4. Select **all** quadrilaterals that **cannot** be cyclic. (Lesson 7-4)

   (A.) a square with side length $\sqrt{5}$ units

   (B.) a 2 inch by 4 inch rectangle

   (C.) a rhombus with side length 5 centimeters and angle measures 20 degrees and 160 degrees

   (D.) quadrilateral *ABCD* in which angle *A* is 62 degrees, angle *B* is 97 degrees, angle *C* is 118 degrees, and angle *D* is 83 degrees

   (E.) quadrilateral *WXYZ* in which angle *W* is 45 degrees, angle *X* is 135 degrees, angle *Y* is 90 degrees, and angle *Z* is 90 degrees

5. A quadrilateral *ABCD* has the given angle measures. Select the set of measurements which could come from a cyclic quadrilateral. (Lesson 7-4)

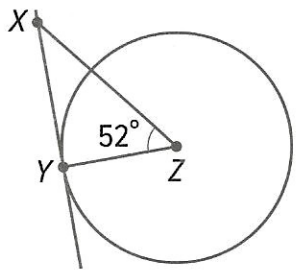
   (A.) angle *A* is 70°, angle *B* is 110°, angle *C* is 70°, and angle *D* is 110°

   (B.) angle *A* is 60°, angle *B* is 50°, angle *C* is 120°, and angle *D* is 130°

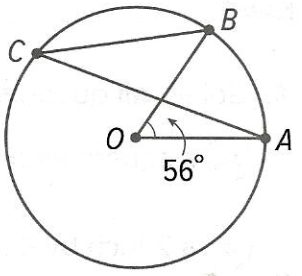
   (C.) angle *A* is 100°, angle *B* is 110°, angle *C* is 70°, and angle *D* is 80°

   (D.) angle *A* is 70°, angle *B* is 45°, angle *C* is 110°, and angle *D* is 45°

6. What is the measure of angle *YXZ*? (Lesson 7-3)



7. The measure of angle AOB is 56 degrees. (Lesson 7-2)



a. What is the measure of angle ACB?

b. What is the measure of the arc from A to B not passing through C?

8. A quadrilateral has vertices $A = (0, 0)$, $B = (2, 4)$, $C = (0, 5)$, and $D = (-2, 1)$. Select the most precise classification for quadrilateral ABCD. (Lesson 6-14)

(A.) quadrilateral

(B.) parallelogram

(C.) rectangle

(D.) square

Lesson 7-6

A Special Point

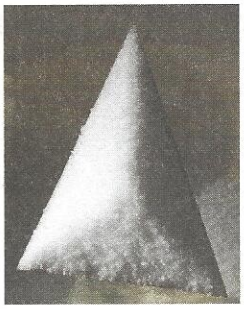
NAME _____ DATE _____ PERIOD _____

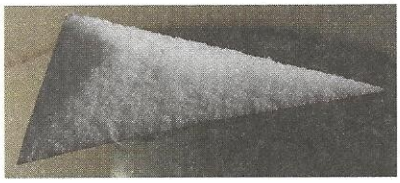
Learning Goal Let's see what we can learn about a triangle by watching how salt piles up on it.

Warm Up

6.1 Notice and Wonder: Salt Pile

What do you notice? What do you wonder?



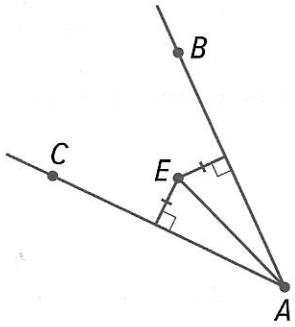


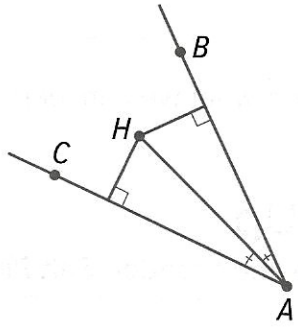
Illustrative Math

Activity

6.2 Point and Angle

Here is an angle *BAC* with 2 different sets of markings.





1. Point *E* is the same distance away from each of the 2 rays that form angle *BAC*. Make a conjecture about angles *EAB* and *EAC* and prove it.

2. Point *H* is on the angle bisector of angle *BAC*. What can you prove about the distances from *H* to each ray?

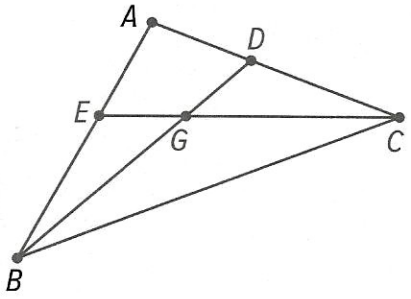
NAME _____ DATE _____ PERIOD _____

Activity

6.3 What If There Are Three Sides?

Two angle bisectors have been constructed in triangle *ABC*. They intersect at point *G*.

1. Sketch segments that show the distances from point *G* to each side of the triangle.



2. How do the distances from point *G* to sides *AB* and *BC* compare? Explain your reasoning.

3. How do the distances from point *G* to sides *AC* and *BC* compare? Explain your reasoning.

4. Will the third angle bisector pass through point *G*? Explain your reasoning.

Are you ready for more?

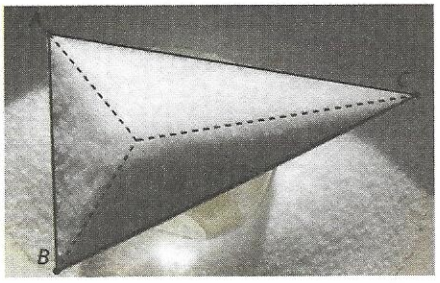
What shape would the ridge form if you poured salt onto a piece of cardboard with a long straight edge and a small hole cut out of it? Explain your reasoning.

NAME _____ DATE _____ PERIOD _____

Summary

A Special Point

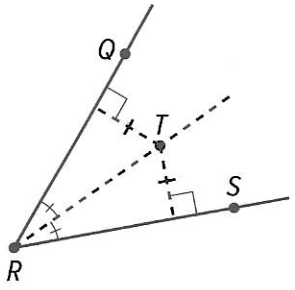
Salt piles up in an interesting way when poured onto a triangle. Why does that happen?

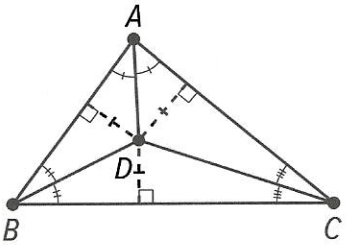


As the salt piles up and reaches a maximum height, new grains of salt will fall off toward whichever side of the triangle is closest. We can show that points on an angle bisector are equidistant from the rays that form the angle. So, salt grains that land on an angle bisector will balance and not fall towards either side. This is why we see ridges form in the salt.

As we might conjecture from the salt example, all 3 angle bisectors in a triangle meet at a single point, called the triangle's **incenter**. To see why this is true, consider any 2 angle bisectors in a triangle. The point where they meet is the same distance from the first and second sides, and also the same distance from the second and third sides. Therefore, it's the same distance from *all* sides, so the third angle bisector must also go through this point.

In the images, segments *RT*, *AD*, *BD*, and *CD* are angle bisectors. This means that point *T* is the same distance from ray *RQ* as it is from ray *RS*. In triangle *ABC*, point *D* is the same distance from all 3 sides of the triangle—it's the triangle's incenter.





Glossary

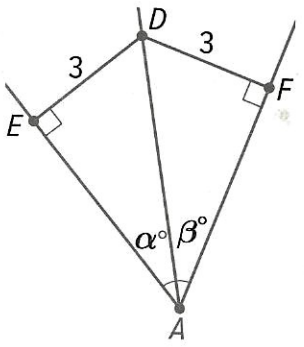
incenter

Illustrative Math

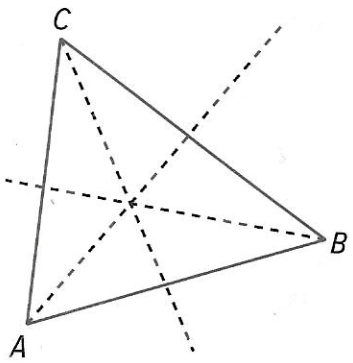
Practice

A Special Point

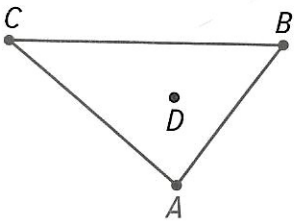
1. How do the values of α and β compare? Explain your reasoning.



2. Triangle *ABC* is shown together with its angle bisectors. Draw a point *D* that is equidistant from sides *AC* and *BC*, but which is closest to side *AB*.

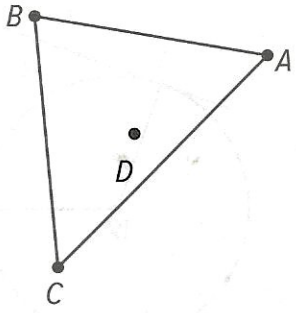


3. In triangle *ABC*, point *D* is the incenter. Sketch segments to represent the distance from point *D* to the sides of the triangle. How must these distances compare?



NAME _____ DATE _____ PERIOD _____

4. Triangle ABC has circumcenter D. (Lesson 7-5)



   a. Sketch the 3 lines that intersect at the circumcenter.

   b. If the distance from point D to point A is 5 units, what is the distance from point D to point C? Explain or show your reasoning.

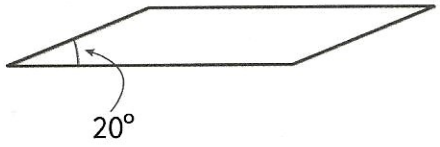
5. The angles of triangle ABC measure 50 degrees, 40 degrees, and 90 degrees. Will its circumcenter fall inside the triangle, on the triangle, or outside the triangle? (Lesson 7-5)

   (A.) inside the triangle

   (B.) on the triangle

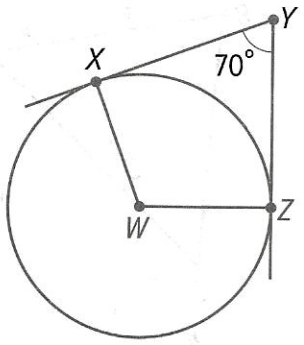
   (C.) outside the triangle

6. Tyler and Kiran are discussing the parallelogram in the image. Tyler says the parallelogram cannot be cyclic. Kiran says the parallelogram can be cyclic if a circle is drawn carefully through the vertices. (Lesson 7-4)



   Do you agree with either of them? Explain or show your reasoning.

7. Find the measures of the remaining angles of quadrilateral *WXYZ*.
(Lesson 7-3)



8. Which expression describes a point that partitions a segment *AB* in a 1 : 5 ratio? (Lesson 6-15)

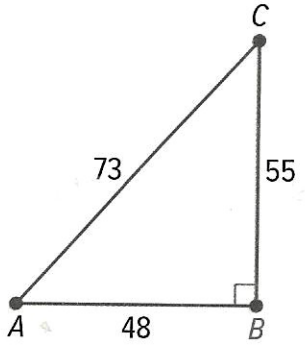
(A) $\frac{1}{5}A + \frac{4}{5}B$

(B) $\frac{1}{6}A + \frac{5}{6}B$

(C) $\frac{4}{5}A + \frac{1}{5}B$

(D) $\frac{5}{6}A + \frac{1}{6}B$

9. Write 3 expressions that can be used to find angle *C*. (Lesson 4-9)



Lesson 7-7

Circles in Triangles

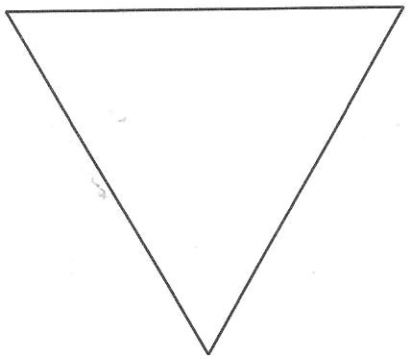
NAME _____ DATE _____ PERIOD _____

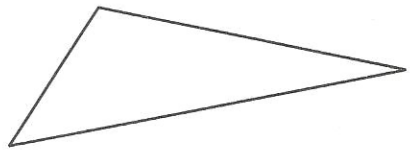
Learning Goal Let's construct the largest possible circle inside of a triangle.

Warm Up

7.1 The Largest Circle

Use a compass to draw the largest circle you can find that fits inside each triangle.





Activity

7.2 The Inner Circle

1. Mark 3 points and connect them with a straightedge to make a large triangle. The triangle should *not* be equilateral.
2. Construct the incenter of the triangle.
3. Construct the segments that show the distance from the incenter to the sides of the triangle.
4. Construct a circle centered at the incenter using one of the segments you just constructed as a radius.
5. Would it matter which of the three segments you use? Explain your thinking.

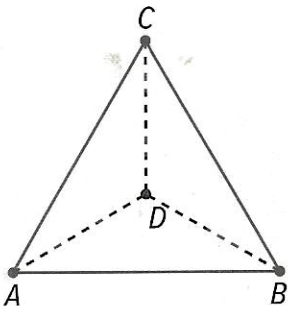
NAME _____ DATE _____ PERIOD _____

Activity

7.3 Equilateral Centers

The image shows an equilateral triangle *ABC*. The angle bisectors are drawn. The incenter is plotted and labeled *D*.

Prove that the incenter is also the circumcenter.



Are you ready for more?

1. Suppose we have an equilateral triangle. Find the ratio of the area of the triangle's circumscribed circle to the area of its inscribed circle.

2. Is this ratio the same for all triangles? Explain or show your reasoning.

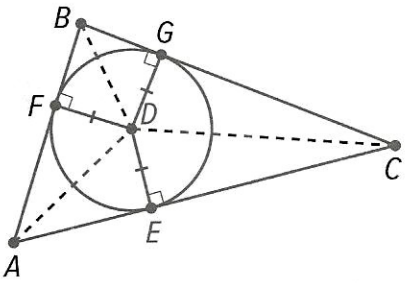
NAME _____ DATE _____ PERIOD _____

Summary

Circles in Triangles

We have seen that the incenter of a triangle is the same distance from all 3 sides of the triangle. If we draw the congruent segments representing the distance from the incenter to the triangle's sides, we can think of them as radii of a circle centered at the incenter. This circle is the triangle's inscribed circle.

In this diagram, segments *BD*, *CD*, and *AD* are angle bisectors. Point *D* is the triangle's incenter, and the circle is inscribed in the triangle.

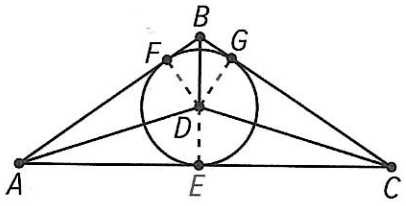


The inscribed circle is the largest possible circle that can be drawn inside a triangle. Also, the 3 radii that represent the distances from the incenter to the sides of the triangle are by definition perpendicular to the sides of the triangle. This means the circle is tangent to all 3 sides of the triangle.

Practice

Circles in Triangles

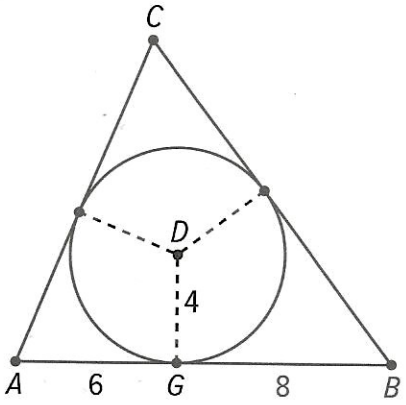
1. Triangle ABC is shown with its incenter at D. The inscribed circle's radius measures 2 units. The length of AB is 9 units. The length of BC is 10 units. The length of AC is 17 units.



   a. What is the area of triangle ACD?

   b. What is the area of triangle ABC?

2. Triangle ABC is shown with an inscribed circle of radius 4 units centered at point D. The inscribed circle is tangent to side AB at the point G. The length of AG is 6 units and the length of BG is 8 units. What is the measure of angle A?



   (A) $\arctan\left(\frac{2}{3}\right)$

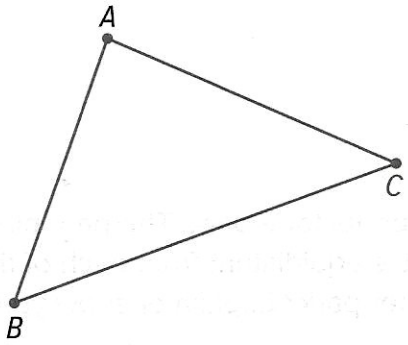
   (B) $2\arctan\left(\frac{2}{3}\right)$

   (C) $\arcsin\left(\frac{2}{3}\right)$

   (D) $2\arccos\left(\frac{2}{3}\right)$

NAME _____ DATE _____ PERIOD _____

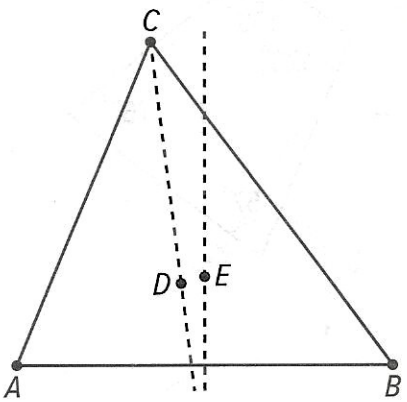
3. Construct the inscribed circle for the triangle.



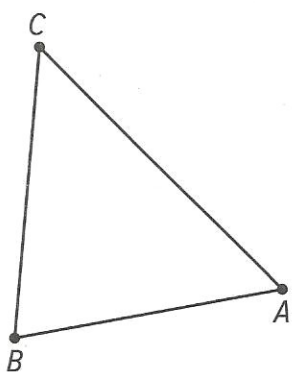
4. Point D lies on the angle bisector of angle ACB. Point E lies on the perpendicular bisector of side AB. (Lesson 7-6)

a. What can we say about the distance between point D and the sides and vertices of triangle ABC?

b. What can we say about the distance between point E and the sides and vertices of triangle ABC?



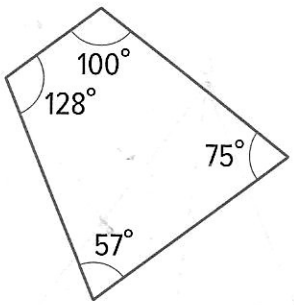
5. Construct the incenter of the triangle. Explain your reasoning. (Lesson 7-6)



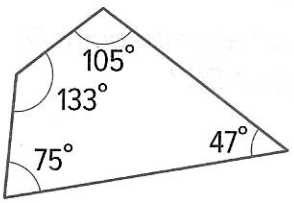
6. The angles of triangle *ABC* measure 30 degrees, 40 degrees, and 110 degrees. Will its circumcenter fall inside the triangle, on the triangle, or outside the triangle? Explain your reasoning. (Lesson 7-5)

7. The images show 2 possible blueprints for a park. The park planners want to build a water fountain that is equidistant from each of the corners of the park. Is this possible for either park? Explain or show your reasoning. (Lesson 7-4)

park A



park B



8. Triangle *ABC* has vertices at (-8, 2), (2, 6), and (10, 2). What is the point of intersection of the triangle's medians? (Lesson 6-16)

Lesson 7-8

Arcs and Sectors

NAME _____ DATE _____ PERIOD _____

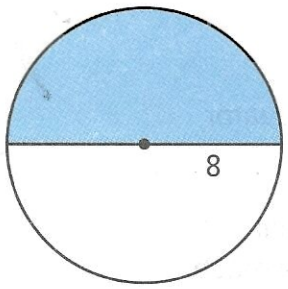
Learning Goal Let's analyze portions of circles.

Warm Up

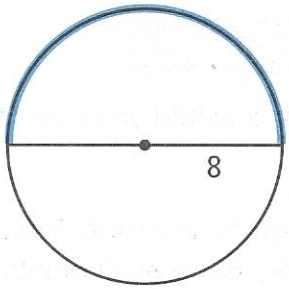
8.1 Math Talk: Fractions of a Circle

Evaluate each problem mentally..

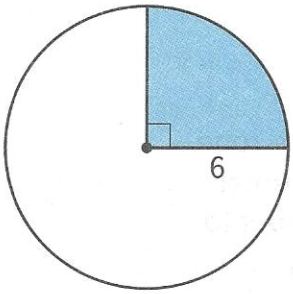
- Find the area of the shaded portion of the circle.



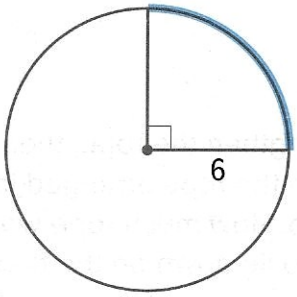
- Find the length of the highlighted portion of the circle's circumference.



- Find the area of the shaded portion of the circle.



- Find the length of the highlighted portion of the circle's circumference.



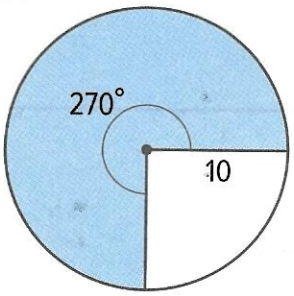
Activity

8.2 Sector Areas and Arc Lengths

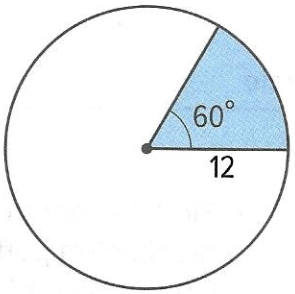
A **sector** of a circle is the region enclosed by 2 radii.

For each circle, find the area of the shaded sector and the length of the arc that outlines the sector. All units are centimeters. Give your answers in terms of π.

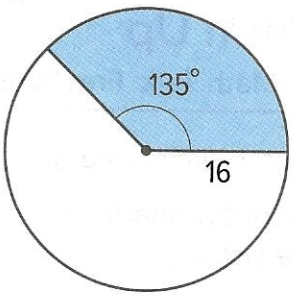
A



B



C



Are you ready for more?

1. What length of rope would you need to wrap around Earth's equator exactly once?

2. Suppose you lengthen the rope, then suspend it so it is an equal distance away from Earth at all points around the equator. How much rope would you have to add in order to allow a person to walk underneath it?

3. What length of rope would you need to wrap around the circumference of a hula hoop with radius 70 centimeters exactly once?

4. Suppose you lengthen the rope, then lay the hula hoop and the rope on the floor, with the rope arranged in a circle with the same center as the hula hoop. How much rope would you have to add in order to allow a person to lie down on the floor with their toes pointing towards the hula hoop and their head towards the rope, but touching neither?

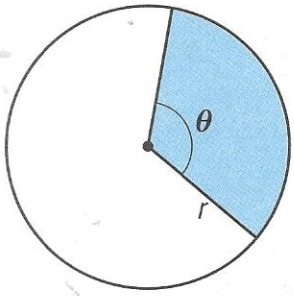
NAME _____ DATE _____ PERIOD _____

Activity

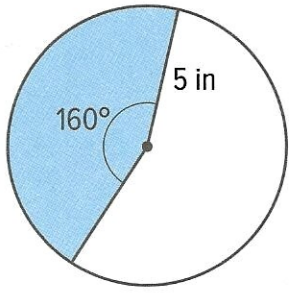
8.3 Build a Method

Mai says, "I know how to find the area of a sector or the length of an arc for central angles like 180 degrees or 90 degrees. But I don't know how to do it for central angles that make up more complicated fractions of the circle."

1. In the diagram, the sector's central angle measures θ degrees and the circle's radius is r units. Use the diagram to tell Mai how to find the *area of a sector* and the *length of an arc* for any angle and radius measure.



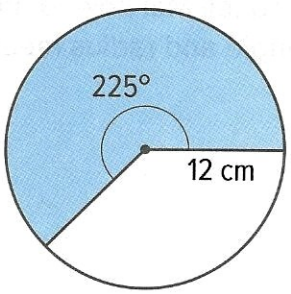
2. This image shows a circle with radius and central angle measurements. Find the area of the shaded sector, and the length of the arc defined by the sector.



Summary

Arcs and Sectors

A **sector** of a circle is the region enclosed by 2 radii. To find the area of a sector, start by calculating the area of the whole circle. Divide the measure of the central angle of the sector by 360 to find the fraction of the circle represented by the sector. Then, multiply this fraction by the circle's total area. We can use a similar process to find the length of the arc lying on the boundary of the sector.



The circle in the image has a total area of 144π square centimeters, and its circumference is 24π centimeters. To find the area of the sector with a 225° central angle, divide 225 by 360 to get $\frac{5}{8}$ or 0.625. Multiply this by 144π to find that the area of the sector is 90π square centimeters. The length of the arc defined by the sector is 15π because $24\pi \cdot \frac{5}{8} = 15\pi$.

Glossary

sector

NAME _____ DATE _____ PERIOD _____

Practice

Arcs and Sectors

1. Suppose a circle is divided into congruent slices. Match each number of slices with the resulting central angle measure of each slice.

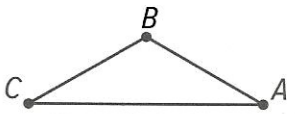
A. 2	1. 45°
B. 3	2. 60°
C. 4	3. 72°
D. 5	4. 120°
E. 6	5. 90°
F. 8	6. 180°

2. A circle of radius 12 units is divided into 8 congruent slices.
   a. What is the area of each slice?

   b. What is the arc length of each slice?

3. Diego says, "To find arc length, divide the measure of the central angle by 360. Then multiply that by the area of the circle." Do you agree with Diego? Show or explain your reasoning.

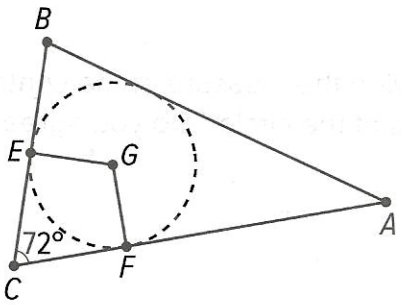
4. The image shows a triangle. (Lesson 7-7)



a. Sketch the inscribed and circumscribed circles for this triangle.

b. Compare and contrast the 2 circles.

5. Triangle *ABC* is shown with its inscribed circle drawn. The measure of angle *ECF* is 72 degrees. What is the measure of angle *EGF*? Explain or show your reasoning. (Lesson 7-7)



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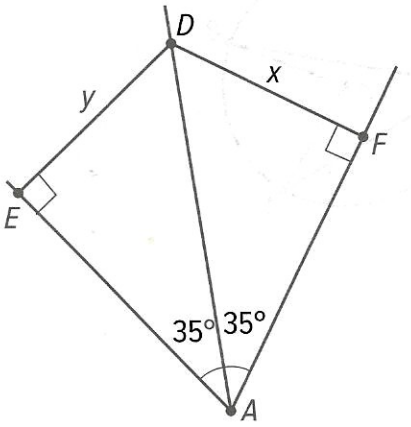
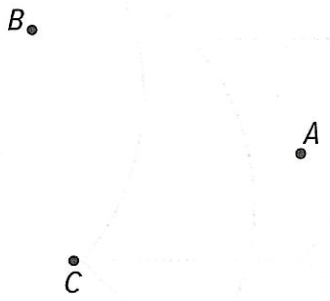
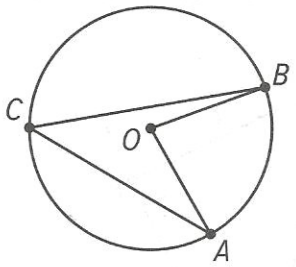
6. How do the values of x and y compare? Explain your reasoning. (Lesson 7-6)



7. Points A, B, and C are the corners of a triangular park. The park district is going to add a set of swings inside the park. The goal is to have the swings equidistant from the vertices of the park. Find a location that meets this goal. Explain or show your reasoning. (Lesson 7-5)



8. In the diagram, the measure of the arc from A to B not passing through C is 80 degrees. What is the measure of angle ACB? **(Lesson 7-2)**



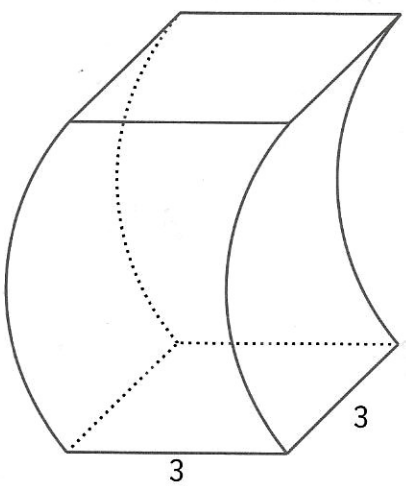
(A.) 20 degrees

(B.) 40 degrees

(C.) 80 degrees

(D.) 160 degrees

9. This solid has curved sides. All cross sections parallel to the base are squares measuring 3 units on each side. The height from the base to the top is 8 units. What is the volume of this solid? **(Lesson 5-10)**



Lesson 7-9

Part to Whole

NAME _____ DATE _____ PERIOD _____

Learning Goal Let's see what we can figure out about a circle if we're given information about a sector of the circle.

Warm Up

9.1 What's Your Angle?

A circle has radius 10 centimeters. Suppose an arc on the circle has length π centimeters. What is the measure of the central angle whose radii define the arc?

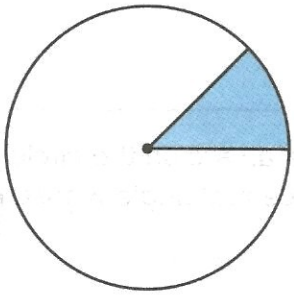
Activity

9.2 Enough Information?

The central angle of this shaded sector measures 45 degrees, and the sector's area is 32π square inches.

Kiran says, "We can find the area of the whole circle, the arc length of the sector, and the circumference of the circle with this information."

Priya says, "But how? We don't know the circle's radius!"



Do you agree with either of them? Explain or show your reasoning. Calculate as many of the values Kiran mentioned as possible.

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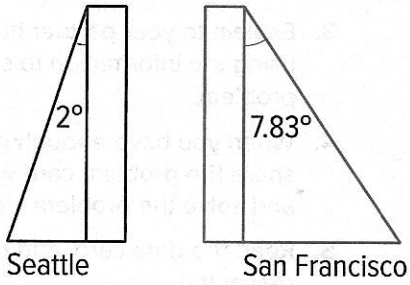
Are you ready for more?

A Greek mathematician named Eratosthenes calculated the circumference of Earth more than 2,000 years ago. He used information similar to what is given here:

Seattle and San Francisco are on the same longitude line and 680 miles apart. At noon on a summer day, an upright meter stick in Seattle casts a shadow with an angle of 2 degrees. On the same day at the same time, an upright meter stick in San Francisco casts a shadow with an angle of 7.83 degrees.

image not to scale



Use this information to calculate the circumference of Earth. Explain or show your reasoning.

Activity

9.3 Info Gap: From Sector to Circle

Your teacher will give you either a problem card or a data card. Do not show or read your card to your partner.

If your teacher gives you the data card:

1. Silently read the information on your card.
2. Ask your partner, "What specific information do you need?" and wait for your partner to ask for information. Only give information that is on your card. (Do not figure out anything for your partner!)
3. Before telling your partner the information, ask, "Why do you need to know (that piece of information)?"
4. Read the problem card, and solve the problem independently.
5. Share the data card, and discuss your reasoning.

If your teacher gives you the problem card:

1. Silently read your card and think about what information you need to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain to your partner how you are using the information to solve the problem.
4. When you have enough information, share the problem card with your partner, and solve the problem independently.
5. Read the data card, and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

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Summary

Part to Whole

We can work backward from information about a sector or arc to get information about the whole circle.

Suppose a circle has an arc with length 7π units that is defined by a 90 degree central angle. This arc makes up $\frac{1}{4}$ of the entire circumference of the circle because $360 \div 90 = 4$. So, we can multiply the arc's length by 4 to find that the entire circumference measures 28π units. The circumference of a circle can be calculated using the expression $2\pi r$, so we know that $2\pi r = 28\pi$. The value of r must be 14 units.

For more difficult problems, we can write equations. Suppose a circle has a sector with area 135π square units and a central angle of 216 degrees. Let x stand for the area of the whole circle. Dividing 216 by 360 gives the fraction of the circle represented by the sector. Multiply that fraction by x and set it equal to the area of the sector.

$$\frac{216}{360}x = 135\pi$$

Solve this equation to find that the circle's area is 225π square units. This tells us that $\pi r^2 = 225\pi$. The value of r must be the positive number that squares to make 225, or 15 units.

Practice

Part to Whole

1. Jada cuts out a rectangular piece of paper that measures 5 inches by 4 inches. Han cuts out a paper sector of a circle with radius 5 inches, and calculates the arc length to be 2π inches. Whose paper is larger? Show your reasoning.

2. A circle has radius 10 centimeters. Suppose an arc on the circle has length 8π centimeters. What is the measure of the central angle whose radii define the arc?

3. A circle has radius 6 units. For each arc length, find the area of a sector of this circle which defines that arc length.

   a. 4π units

   b. 5π units

   c. 10 units

   d. ℓ units

4. Select **all** the sectors which have an area of 3π square units. (Lesson 7-8)

   (A.) a sector with a radius of 6 units and a central angle of 30 degrees

   (B.) a sector with a radius of 6 units and a central angle of 45 degrees

   (C.) a sector with a radius of 3 units and a central angle of 60 degrees

   (D.) a sector with a radius of 3 units and a central angle of 120 degrees

   (E.) a sector with a radius of 3 units and a central angle of 180 degrees

5. A circle has radius 4 units and a central angle measuring 45 degrees. What is the length of the arc defined by the central angle? (Lesson 7-8)

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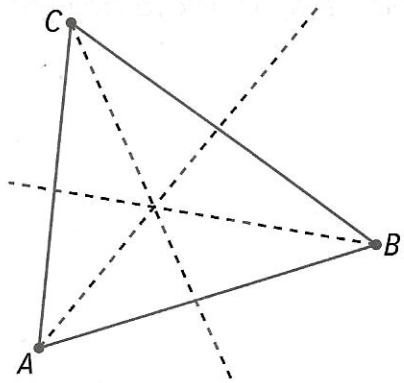
6. Clare and Diego are discussing inscribing circles in quadrilaterals. **(Lesson 7-7)**

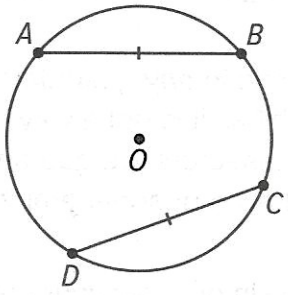
Diego thinks that you can inscribe a circle in *any* quadrilateral since you can inscribe a circle in any triangle. Clare thinks it is not always possible because she does not think the angle bisectors are guaranteed to intersect at a single point. She claims she can draw a quadrilateral for which an inscribed circle can't be drawn.

Do you agree with either of them? Explain or show your reasoning.

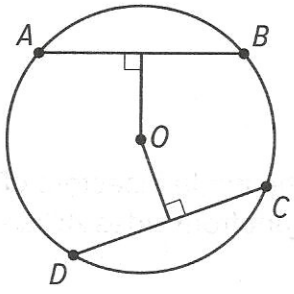
7. Triangle *ABC* is shown together with the angle bisectors of each of its angles. Draw a point *D* that is equidistant from sides *AC* and *AB*, but which is closest to side *BC*. **(Lesson 7-6)**



8. Priya and Mai are trying to prove that if 2 chords are congruent, they are equidistant from the center of the circle. Priya draws this picture. (Lesson 7-1)



Mai adds the perpendicular segment from the center of the circle to each chord.



Priya says, "I think we should try to use triangles because that is how we proved things congruent before." Mai says, "I think you're right, but how? Should we draw in some radii?"

Help them complete the proof.

Lesson 7-10

Angles, Arcs, and Radii

NAME _____ DATE _____ PERIOD _____

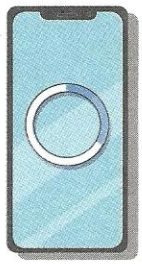
Learning Goal Let's analyze relationships between arc lengths, radii, and central angles.

Warm Up

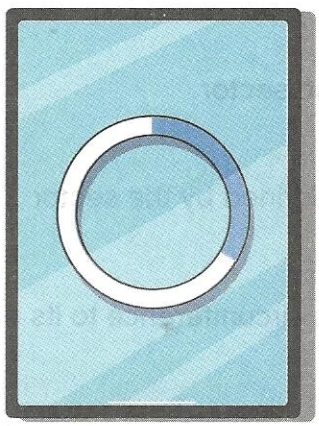
10.1 Comparing Progress

Han and Tyler are each completing the same set of tasks on an online homework site. Han is using his smartphone and Tyler is using his tablet computer. Their progress is indicated by the circular bars shown in the image. The shaded arc represents the tasks that have been completed. When the full circumference of the circle is shaded, they will be finished with all the tasks.

Han's progress



Tyler's progress



Tyler says, "The arc length on my progress bar measures 4.75 centimeters. The arc length on Han's progress bar measures 2.25 centimeters. So, I've completed more tasks than Han has."

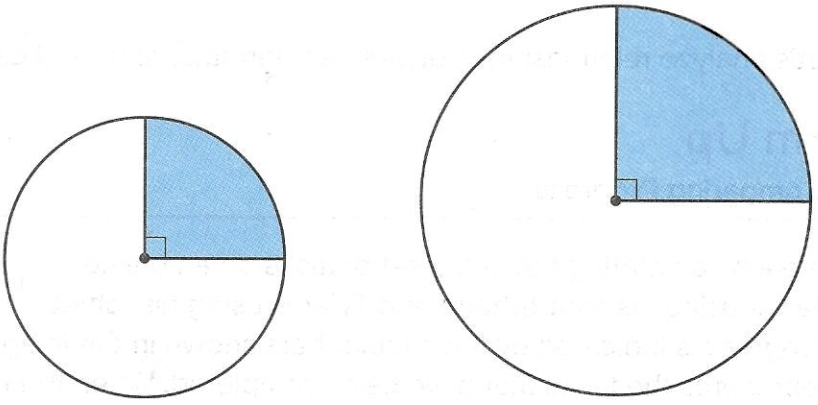
1. Do you agree with Tyler? Why or why not?

2. What information would you need to make a completely accurate comparison between the two students' progress?

Activity

10.2 A Dilated Circle

The image shows 2 circles. The second circle is a dilation of the first circle using a scale factor of 3.



For each part of the dilated image, determine the factor by which it's changed when compared to the corresponding part of the original circle.

1. the area of the sector
2. the central angle of the sector
3. the radius
4. the length of the arc defined by the sector

5. the ratio of the circle's circumference to its diameter

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Activity

10.3 Card Sort: Angles, Arcs, and Radii

Your teacher will give you a set of cards. Sort the cards into categories of your choosing. Be prepared to explain the meaning of your categories. Then, sort the cards into categories in a different way. Be prepared to explain the meaning of your new categories.

Are you ready for more?

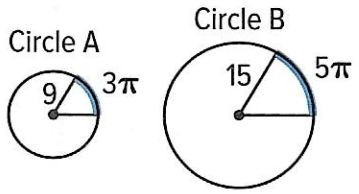
For a circle of radius r, an expression that relates the area of a sector to the arc length defined by that sector is $A = \frac{1}{2} r\ell$ where A is the area of the sector and ℓ is the length of the arc. Explain why this is true and provide an example.

Summary

Angles, Arcs, and Radii

If we have the same central angle in 2 different circles, the length of the arc defined by the angle depends on the size of the circle. So, we can use the relationship between the arc length and the circle's radius to get some information about the size of the central angle.

For example, suppose circle A has radius 9 units and a central angle that defines an arc with length 3π. Circle B has radius 15 units and a central angle that defines an arc with length 5π. How do the 2 angles compare?



For the angle in Circle A, the ratio of the arc length to the radius is $\frac{3\pi}{9}$, which can be rewritten as $\frac{\pi}{3}$. For the angle in Circle B, the arc length to radius ratio is $\frac{5\pi}{15}$, which can also be written as $\frac{\pi}{3}$. That seems to indicate that the angles are the same size. Let's check.

Circle A's circumference is 18π units. The arc length 3π is $\frac{1}{6}$ of 18π, so the angle measurement must be $\frac{1}{6}$ of 360 degrees, or 60 degrees. Circle B's circumference is 30π units. The arc length 5π is $\frac{1}{6}$ of 30π, so this angle also measures $\frac{1}{6}$ of 360 degrees or 60 degrees. The 2 angles are indeed congruent.

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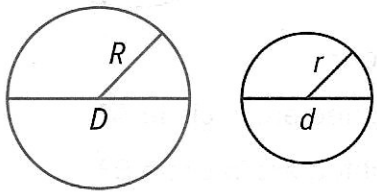
Practice

Angles, Arcs, and Radii

1. Here are 2 circles. The smaller circle has radius r, circumference c, and diameter d. The larger circle has radius R, circumference C, and diameter D. The larger circle is a dilation of the smaller circle by a factor of k.

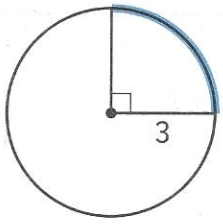


Using the circles, write 3 pairs of equivalent ratios. Find the value of each set of ratios you wrote.

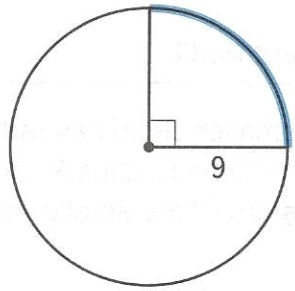
2. Tyler is confident that all circles are similar, but he cannot explain why this is true. Help Tyler explain why all circles are similar.

3. Circle B is a dilation of circle A.

circle A



circle B



a. What is the scale factor of dilation?

b. What is the length of the highlighted arc in circle A?

c. What is the length of the highlighted arc in circle B?

d. What is the ratio of the arc lengths?

e. How does the ratio of arc length compare to the scale factor?

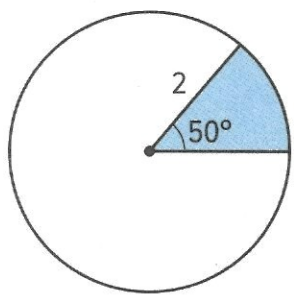
4. Kiran cuts out a square piece of paper with side length 6 inches. Mai cuts out a paper sector of a circle with radius 6 inches, and calculates the arc length to be 4π inches. Whose paper is larger? Explain or show your reasoning. (Lesson 7-9)

5. A circle has radius 3 centimeters. Suppose an arc on the circle has length 4π centimeters. What is the measure of the central angle whose radii define the arc? (Lesson 7-9)

6. A circle with a shaded sector is shown. (Lesson 7-8)

a. What is the area of the shaded sector?

b. What is the length of the arc that outlines this sector?



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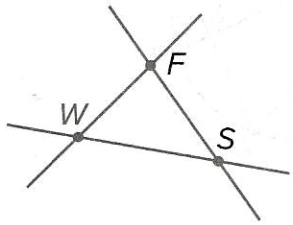
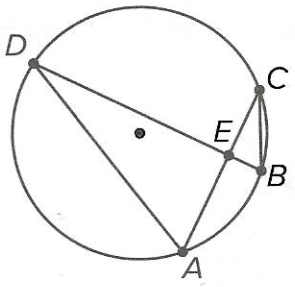
7. The towns of Washington, Franklin, and Springfield are connected by straight roads. The towns wish to build an airport to be shared by all of them. **(Lesson 7-7)**



a. Where should they build the airport if they want it to be the same distance from each town's center? Describe how to find the precise location.

b. Where should they build the airport if they want it to be the same distance from each of the roads connecting the towns? Describe how to find the precise location.

8. Chords AC and DB intersect at point E. Select **all** pairs of angles that must be congruent. (Lesson 7-2)



(A.) angle ADB and angle ACB

(B.) angle ADB and angle CAD

(C.) angle DEA and angle CEB

(D.) angle CAD and angle CBD

(E.) angle BCA and angle CBD

Lesson 7-11

A New Way to Measure Angles

NAME _____ DATE _____ PERIOD _____

Learning Goal Let's look at a new way to measure angles.

Warm Up

11.1 A One-Unit Radius

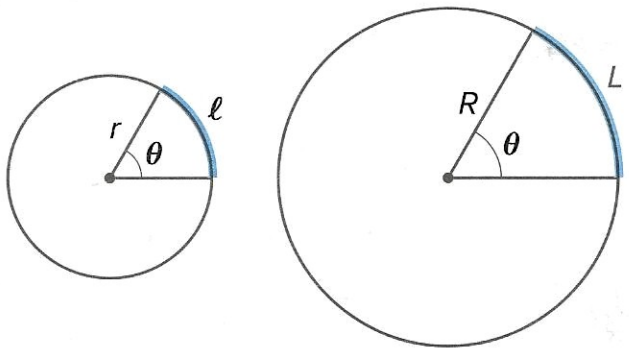
A circle has radius 1 unit. Find the length of the arc defined by each of these central angles. Give your answers in terms of π.

1. 180 degrees
2. 45 degrees
3. 270 degrees
4. 225 degrees
5. 360 degrees

Activity

11.2 A Constant Ratio

Diego and Lin are looking at 2 circles.



Diego says, "It seems like for a given central angle, the arc length is proportional to the radius. That is, the ratio $\frac{\ell}{r}$ has the same value as the ratio $\frac{L}{R}$ because they have the same central angle measure. Can we prove that this is true?"

Lin says, "The big circle is a dilation of the small circle. If k is the scale factor, then $R = kr$."

Diego says, "The arc length in the small circle is $\ell = \frac{\theta}{360} \cdot 2\pi r$. In the large circle, it's $L = \frac{\theta}{360} \cdot 2\pi R$. We can rewrite that as $L = \frac{\theta}{360} \cdot 2\pi rk$. So $L = k\ell$."

Lin says, "Okay, from here I can show that $\frac{\ell}{r}$ and $\frac{L}{R}$ are equivalent."

1. How does Lin know that the big circle is a dilation of the small circle?

2. How does Lin know that $R = kr$?

3. Why could Diego write $\ell = \frac{\theta}{360} \cdot 2\pi r$?

4. When Diego says that $L = k\ell$, what does that mean in words?

5. Why could Diego say that $L = k\ell$?

6. How can Lin show that $\frac{\ell}{r} = \frac{L}{R}$?

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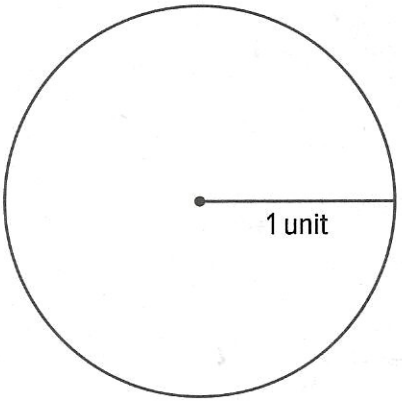
Activity

11.3 Defining Radians

Suppose we have a circle that has a central angle. The **radian** measure of the angle is the ratio of the length of the arc defined by the angle to the circle's radius. That is, $\theta = \frac{\text{arc length}}{\text{radius}}$.

1. The image shows a circle with radius 1 unit.



   a. Cut a piece of string that is the length of the radius of this circle.

   b. Use the string to mark an arc on the circle that is the same length as the radius.

   c. Draw the central angle defined by the arc.

   d. Use the definition of radian to calculate the radian measure of the central angle you drew.

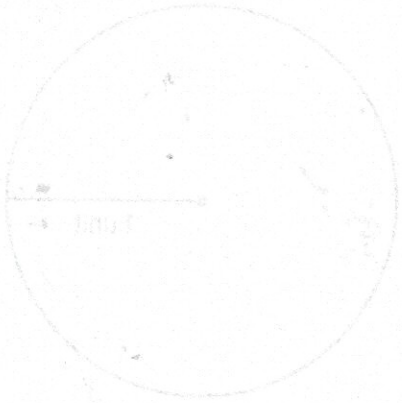
2. Draw a 180 degree central angle (a diameter) in the circle. Use your 1-unit piece of string to measure the approximate length of the arc defined by this angle.

3. Calculate the radian measure of the 180 degree angle. Give your answer both in terms of π and as a decimal rounded to the nearest hundredth.

4. Calculate the radian measure of a 360 degree angle.

Are you ready for more?

Research where the "360" in 360 degrees comes from. Why did people choose to define a degree as $\frac{1}{360}$ of the circumference of a circle?



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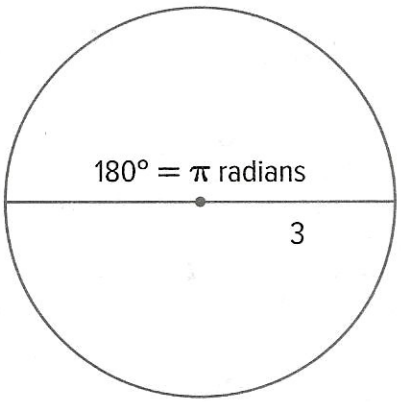
Summary

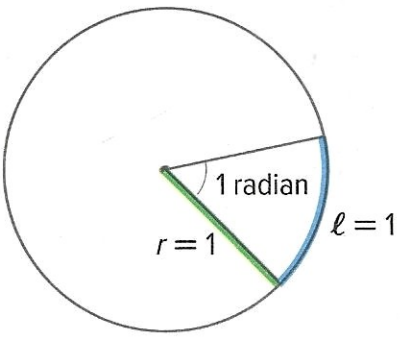
A New Way to Measure Angles

Degrees are one way to measure the size of an angle. **Radians** are another way to measure angles. Assume an angle's vertex is the center of a circle. The radian measure of the angle is the ratio between the length of the arc defined by the angle and the radius of the circle. We can write this as $\theta = \frac{\text{arc length}}{\text{radius}}$. This ratio is constant for a given angle, no matter the size of the circle.

Consider a 180 degree central angle in a circle with radius 3 units. The arc length defined by the angle is 3π units. The radian measure of the angle is the ratio of the arc length to the radius, which is π radians because $\frac{3\pi}{3} = \pi$.



Another way to think of the radian measure of the angle is that it measures the number of radii that would make up the length of the arc defined by the angle. For example, if we draw an arc that is the same length as the radius, both the arc length and the radius are 1 unit. The radian measure of the central angle that defines this arc is the quotient of those values, or 1 radian.



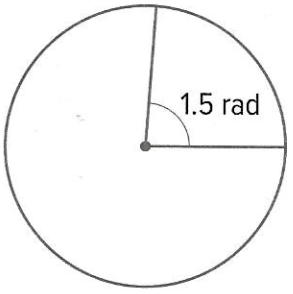
Glossary

radian

Practice

A New Way to Measure Angles

1. Here is a central angle that measures 1.5 radians. Select **all** true statements.



A. The radius is 1.5 times longer than the length of the arc defined by the angle.

B. The length of the arc defined by the angle is 1.5 times longer than the radius.

C. The ratio of arc length to radius is 1.5.

D. The ratio of radius to arc length is 1.5.

E. The area of the whole circle is 1.5 times the area of the slice.

F. The circumference of the whole circle is 1.5 times the length of the arc formed by the angle.

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2. Match each arc length ℓ and radius r with the measure of the central angle of the arc in radians.

(A.) $r = 2, \ell = \frac{\pi}{2}$

(B.) $r = 3, \ell = 2$

(C.) $r = 3.5, \ell = 2.8$

(D.) $r = 4, \ell = 3$

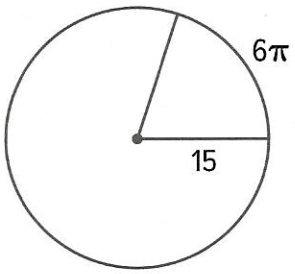
(E.) $r = 5, \ell = \pi$

(F.) $r = 6, \ell = 4$

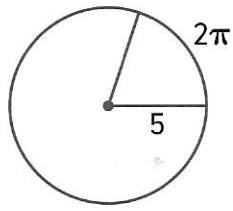
1. $\frac{\pi}{5}$ radian
2. $\frac{2}{3}$ radians
3. 0.75 radians
4. $\frac{\pi}{4}$ radians
5. 0.8 radians

3. Han thinks that since the arc length in circle A is longer, its central angle is larger. Do you agree with Han? Show or explain your reasoning.

circle A

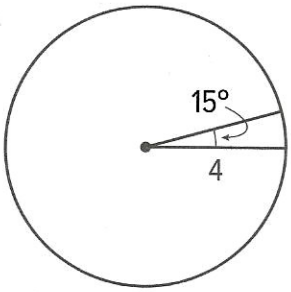


circle B

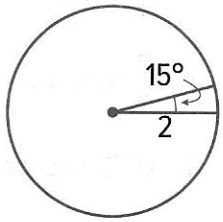


4. Circle B is a dilation of circle A. (Lesson 7-10)

circle A



circle B



a. What is the scale factor?

b. What is the area of the 15 degree sector in circle A?

c. What is the area of the 15 degree sector in circle B?

d. What is the ratio of the areas of the sectors?

e. How does the ratio of areas of the sectors compare to the scale factor?

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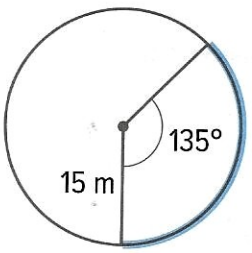
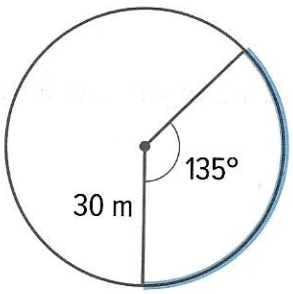
5. Priya and Noah are riding different size Ferris wheels at a carnival. They started at the same time. The highlighted arcs show how far they have traveled. (Lesson 7-10)

Noah's Ferris wheel



Priya's Ferris wheel



a. How far has Noah traveled?

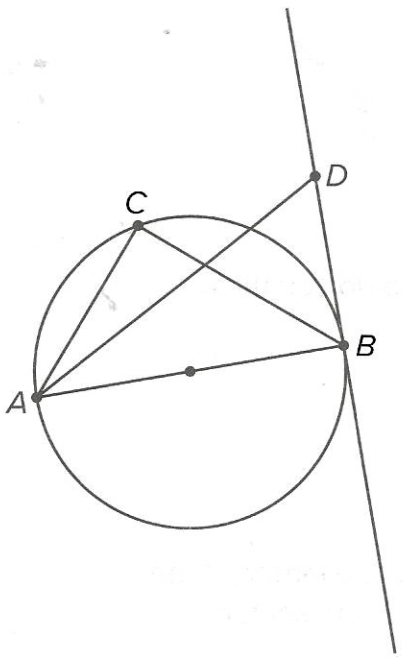
b. How far has Priya traveled?

c. If the Ferris wheels will each complete 1 revolution, who do you think will finish first?

6. A circle has radius 8 units, and a central angle is drawn in. The length of the arc defined by the central angle is 4π units. Find the area of the sector outlined by this arc. (Lesson 7-9)

7. Clare is trying to explain how to find the area of a sector of a circle. She says, "First, you find the area of the whole circle. Then, you divide by the radius." Do you agree with Clare? Explain or show your reasoning. (Lesson 7-8)

8. Line BD is tangent to a circle with diameter AB. List 2 right angles. (Lesson 7-3)



Lesson 7-12

Radian Sense

NAME _____ DATE _____ PERIOD _____

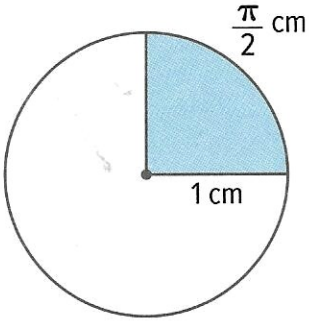
Learning Goal Let's get a sense for the sizes of angles measured in radians.

Warm Up

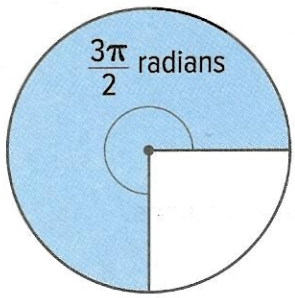
12.1 Which One Doesn't Belong: Angle Measures

Which one doesn't belong?

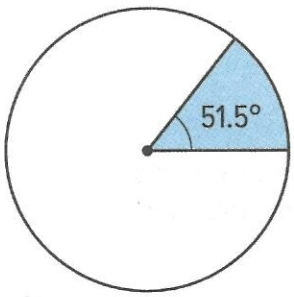
A



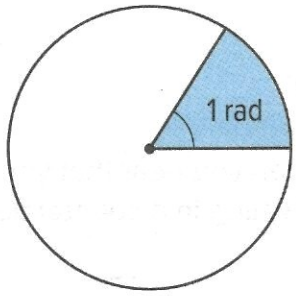
B



C



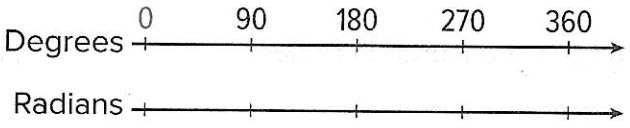
D



Activity

12.2 Degrees Versus Radians

This double number line shows degree measurements on one line and radians on another.



1. Fill in the radian measures on the bottom line for 0°, 90°, 180°, 270°, and 360°.

2. Express each radian measurement in degrees.

   a. $\frac{\pi}{3}$ radians

   b. $\frac{5\pi}{4}$ radians

3. Express each degree measurement in radians.

   a. 30°

   b. 120°

Are you ready for more?

Your boat is heading due south when you hear that you must head north-east to return home. You turn the boat, traveling in a counterclockwise circular path until you're facing north-east.

1. Sketch the path of the boat.

2. If the circle you traced had a radius of 40 feet, what distance did you travel?

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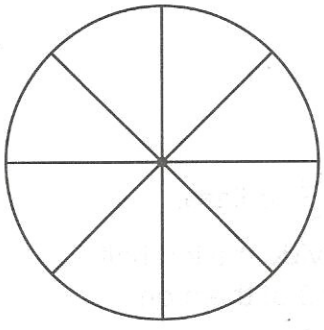
Activity

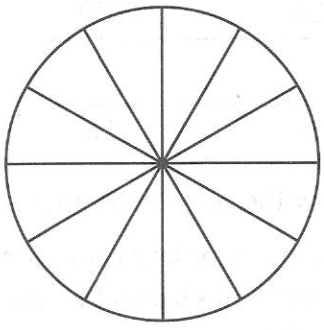
12.3 Pie Coloring Contest

Your teacher will give you a set of cards with angle measures on them. Place the cards upside down in a pile. Choose 1 student to go first. This student should draw a card, then on either circle shade a sector of the circle whose central angle is the measure on the card that was drawn.

Take turns repeating these steps. If you are shading in a circle that already has a shaded sector, choose a spot next to the already-shaded sectors—don't leave any gaps. You might have to draw additional lines to break the sectors into smaller pieces.

Continue until an angle is drawn that won't fit in any of the sectors that are still blank.





When you're finished, answer these questions about each circle:

1. What is the central angle measure for the remaining unshaded sector?

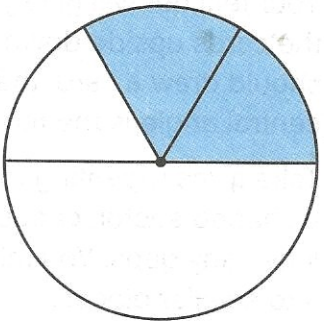
2. What is the central angle measure for the block of shaded sectors?

Summary

Radian Sense

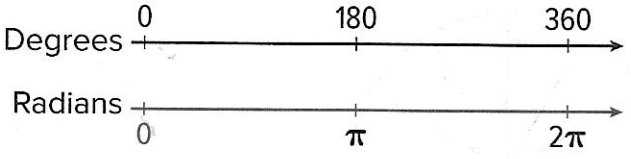
We can divide circles into congruent sectors to get a sense for the size of an angle measured in radians.

Suppose we want to draw an angle that measures $\frac{2\pi}{3}$ radians.



We know that π radians is equivalent to 180 degrees. If we divide a sector with a central angle of π radians into thirds, we can shade in 2 of them to create an angle measuring $\frac{2\pi}{3}$ radians.

Another way to understand the size of an angle measured in radians is to create a double number line with degrees on one line and radians on the other. On the double number line in the image, the degree measurements are aligned with their equivalent radian measures. For example, π radians is equivalent to 180°.



Suppose we need to know the size of an angle that measures $\frac{3\pi}{4}$ radians.

The left half of the double number line represents π radians. Divide the left half of the top and bottom number lines into fourths, then count out 3 of them on the radians line to land on $\frac{3\pi}{4}$. On the top line, each interval we drew represents 45 degrees because $180 \div 4 = 45$. If we count 3 of those intervals, we find that $\frac{3\pi}{4}$ radians is equivalent to 135 degrees because $45 \cdot 3 = 135$.

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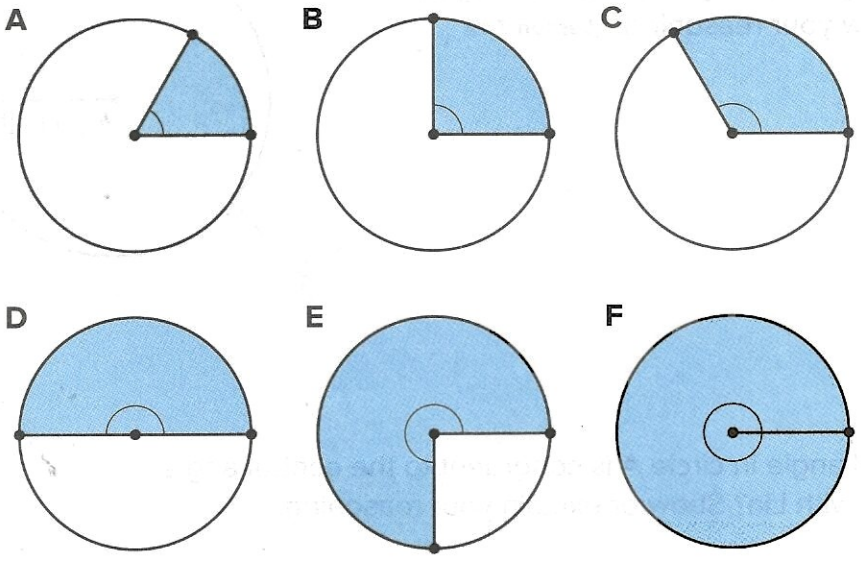
Practice

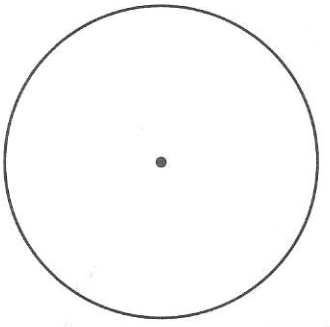
Radian Sense

1. Match each diagram showing a sector with the measure of its central angle in radians.

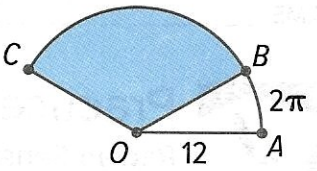


A. diagram A	1. $\frac{3\pi}{2}$
B. diagram B	2. π
C. diagram C	3. $\frac{\pi}{3}$
D. diagram D	4. $\frac{\pi}{2}$
E. diagram E	5. 2π
F. diagram F	6. $\frac{2\pi}{3}$

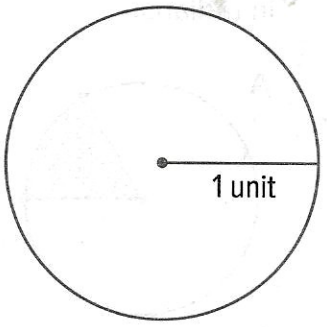
2. In the circle, sketch a central angle that measures $\frac{2\pi}{3}$ radians.



3. Angle AOC has a measure of $\frac{5\pi}{6}$ radians. The length of arc AB is 2π units and the radius is 12 units. What is the area of sector BOC?

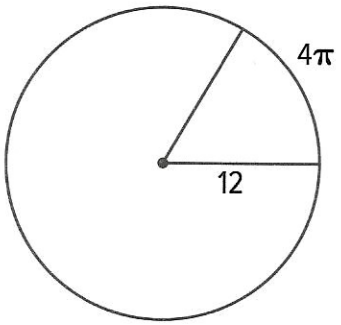


4. Calculate the radian measure of a 30 degree angle. Use any method you like, including sketching in the circle diagram provided. Explain or show your reasoning. (Lesson 7-11)

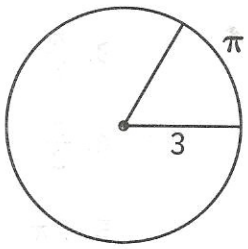


5. Lin thinks that the central angle in circle A is congruent to the central angle in circle B. Do you agree with Lin? Show or explain your reasoning. (Lesson 7-11)

circle A



circle B



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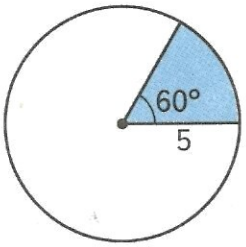
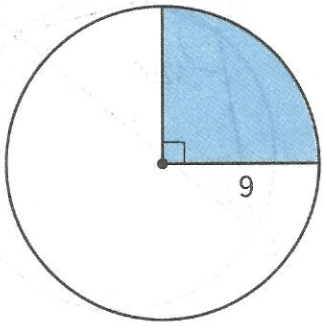
6. **circle A**



circle B



Select **all** true statements. (Lesson 7-10)

(A.) The sector in circle B has a larger area than the sector in circle A.

(B.) Not taking into account the sectors, circle A and circle B are similar.

(C.) The fraction of the circumference taken up by the arc outlining circle A's sector is smaller than the fraction of the circumference taken up by the arc in circle B.

(D.) The ratio of the area of circle A's sector to its total area is $\frac{1}{6}$.

(E.) The ratio of circle A's area to circle B's area is $\frac{5}{9}$.

7. Match each arc length and radius with the measure of the central angle that defines the arc. (Lesson 7-9)

A. arc length 5π cm, radius 5 cm

B. arc length 4π cm, radius 10 cm

C. arc length 9π cm, radius 12 cm

D. arc length 3π cm, radius 18 cm

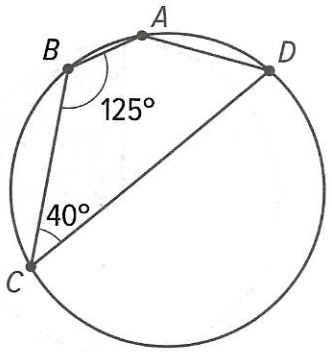
1. angle: 30 degrees

2. angle: 72 degrees

3. angle: 135 degrees

4. angle: 180 degrees

8. Quadrilateral *ABCD* is shown with the given angle measures. Select **all** true statements. (Lesson 7-4)



(A.) Angle *A* measures 140 degrees.

(B.) The measures of angle *A* and angle *D* must add to 180 degrees.

(C.) Angle *A* measures 55 degrees.

(D.) Angle *D* measures 55 degrees.

(E.) Angle *D* measures 40 degrees.

Lesson 7-13

Using Radians

NAME _____ DATE _____ PERIOD _____

Learning Goal Let's see how radians can help us calculate sector areas and arc lengths.

Warm Up

13.1 What Fraction?

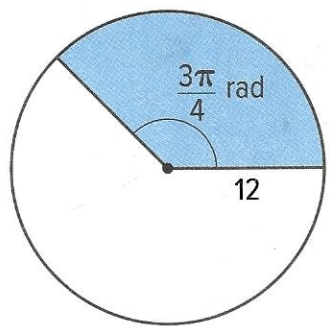
A circle with radius 24 inches has a sector with central angle $\frac{\pi}{3}$ radians.

1. What fraction of the whole circle is represented by this sector?
2. Find the area of the sector.

Activity

13.2 A Sector Area Shortcut

Lin and Elena are trying to find the area of the shaded sector in the image. Lin says, "We've found sector areas when the central angle is given in degrees, but here it's in radians. Should I start by finding the area of the full circle?"



Elena says, "I saw someone using the formula $\frac{1}{2} r^2 \theta$ where θ is the measure of the angle in radians, and r is the radius. But I don't know where that came from."

1. Compare and contrast finding sector areas for central angles measured in degrees and those measured in radians.

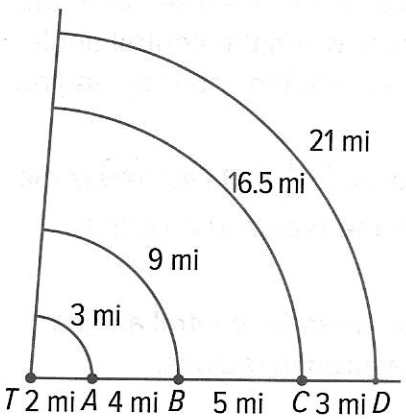
2. Explain why the formula that Elena saw works.

3. Find the area of the sector.

Activity

13.3 An Arc Length Shortcut

The city of Riverside has 2 straight highways extending out from its town hall (located at T in the image). Several roads shaped like circular arcs connect the two highways. Exits from 1 of the highways are shown at points A, B, C, and D.



1. Create a table that shows the arc length, ℓ, of the connector roads as a function of the radius, r, of the highway .

r (Miles)	ℓ (Miles)

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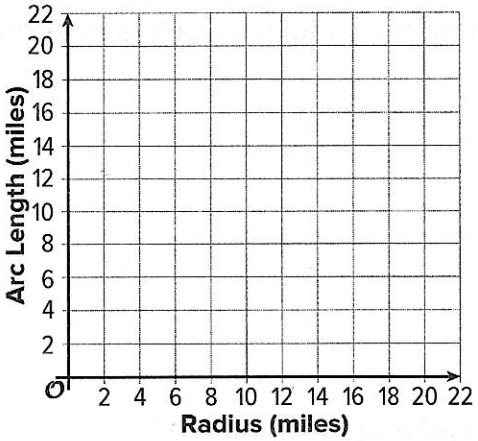
2. Plot the points from your table on the coordinate grid and connect them.



3. The points should form a line. Write an equation for this line, using the variables ℓ and r.

4. What does the slope of the line mean in context of the highways and connector roads?

5. The city wants to build a new arc-shaped connector road starting at point E which will be an additional 6 miles past exit D. How long will this road be?

Are you ready for more?

Suppose a Ferris wheel with a 50 foot radius takes 2 minutes to complete 1 rotation.

1. After 20 seconds, through how many radians would a rider on the Ferris wheel have traveled?

2. How long would it take to travel through 3 radians?

3. What is the rotational speed of the rider on the ferris wheel in radians per second?

4. A decorative light is attached to the Ferris wheel 25 feet from the center of the wheel. What is the rotational speed of the light in radians per second?

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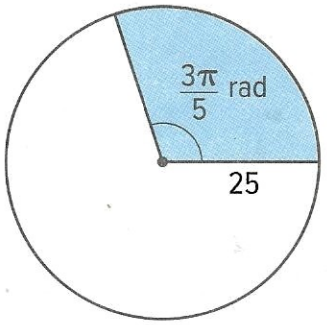
Summary

Using Radians

Suppose we want to find the area of a sector of a circle whose central angle is θ radians and whose radius is r units. First we find the area of the whole circle, or πr^2 square units. Then we find the fraction of the circle represented by the sector. The complete circle measures 2π radians. So the fraction is $\frac{\theta}{2\pi}$. Multiply the fraction by the area to get $\frac{\theta}{2\pi} \cdot \pi r^2$. This can be rewritten as $\frac{1}{2} r^2 \theta$.

For example, let the radius be 25 units and the radian measure of the sector's central angle be $\frac{3\pi}{5}$ radians. Substitute these values into the formula we just created to get $\frac{1}{2}(25)^2 \cdot \frac{3\pi}{5}$. This can be rewritten as $\frac{375\pi}{2}$, which is about 589 square units. Using the formula was a bit quicker than going through the process of finding the specific fraction of the circle represented by this sector and multiplying by the circle's area.



Additionally, we can write a formula for arc length based on the definition of radian measure as $\theta = \frac{\ell}{r}$ where θ is the central angle measure in radians, ℓ is the arc length, and r is the radius. Rewrite the definition to solve for arc length. We get $\ell = r\theta$.

Practice

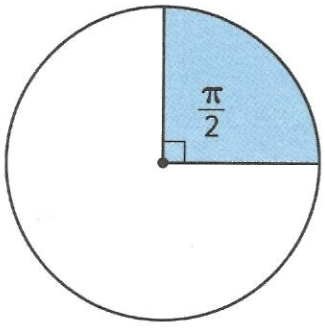
Using Radians

1. Respond to each question
   a. Find the arc length of an arc with a central angle of $\frac{2\pi}{3}$ radians and a radius of 6 units.
   b. Find the arc length of an arc with a central angle of $\frac{5\pi}{6}$ radians and a circumference of 12 units.
   c. Find the measure (in radians) of a central angle of an arc with arc length 4 units and radius 3 units.

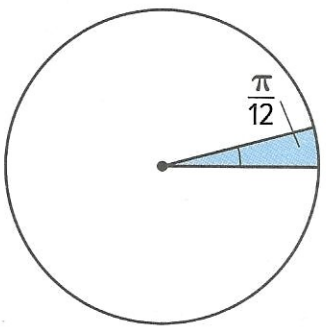
2. A circle has radius 10 units. For each angle measure, find the area of a sector of this circle with that central angle.
   a. $\frac{\pi}{8}$ radians
   b. $\frac{3\pi}{4}$ radians
   c. 1 radian
   d. 2 radians

3. Each circle has a shaded sector with a central angle measured in radians. What fraction of the circle is each sector?

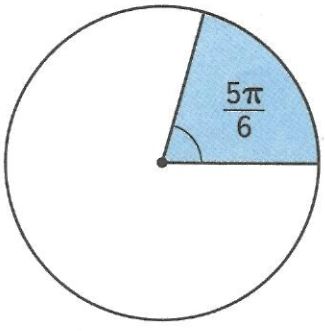
circle A



circle B



circle C



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4. Find the radian measure of each angle. **(Lesson 7-12)**

a. 30 degrees

b. 45 degrees

c. 50 degrees

5. Find the degree measure of each angle. **(Lesson 7-12)**

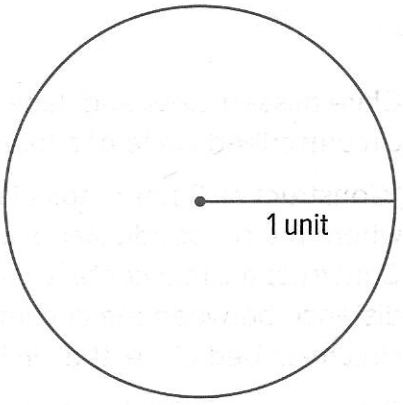
a. $\frac{\pi}{3}$ radians

b. $\frac{\pi}{2}$ radians

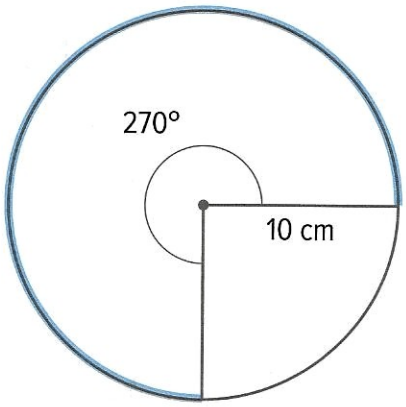
c. $\frac{3\pi}{4}$ radians

d. 3 radians

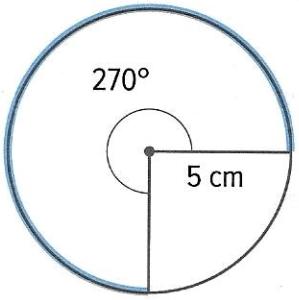
6. Calculate the radian measure of a 225 degree angle. Use any method you like, including sketching in the circle diagram provided. Explain or show your reasoning. **(Lesson 7-11)**



7. **Andre's circle**



Diego's circle



Andre and Diego are each saving money. They each hope to save 500 dollars. They are tracking their progress on the circles in the image.

Diego thinks that Andre has saved more money than Diego has saved. Andre thinks they have saved the same amount. Do you agree with either of them? Explain or show your reasoning. (Lesson 7-10)

8. Clare missed class and Jada is teaching her how to construct the circumscribed circle of a triangle. Here are the instructions Jada wrote.

"Construct all 3 perpendicular bisectors of the triangle's sides. The point where the perpendicular bisectors intersect is called the circumcenter. Construct a circle centered at the circumcenter with radius set to the distance between the circumcenter and a vertex. If the triangle has a circumscribed circle, the circle you construct will go through all 3 vertices."

Do you agree with Jada's instructions? Explain your reasoning. (Lesson 7-5)

Lesson 7-14

Putting It All Together

NAME _____ DATE _____ PERIOD _____

Learning Goal Let's use geometry to solve problems.

Warm Up

14.1 Equal Slices

At a pizza restaurant, a personal pizza has a radius of 10 centimeters and costs $5. Another restaurant takes a pizza with radius 30 centimeters, cuts it into 8 slices of equal area, and charges $5 per slice. Which is a better deal? Explain your reasoning.

Activity

14.2 Pizza Palooza

Elena was researching offers for the upcoming Pizza Palooza festival. She wants to get a good deal on a single slice of pizza.

Your teacher will give you cards that show the deals offered by 4 vendors. Which vendor should Elena choose? Explain or show your reasoning.

Activity

14.3 A Fair Split

Jada and Andre want to share a big slice of pizza so that each of them gets the same amount, but Andre doesn't like the crust. The pizza slice is a sector of a circle with a radius of 20 cm and a central angle that measures $\frac{\pi}{3}$ radians.

How can Andre and Jada divide the slice of pizza into 2 equal pieces so that Andre doesn't have to eat any crust?

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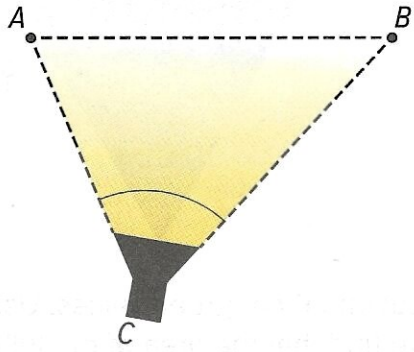
Activity

14.4 Let Your Light Shine

Noah is taking photos of a sculpture he made in art class. He will submit the photos to a contest. The sculpture is in front of a backdrop, which is represented from an overhead view in the image by segment *AB*. Noah positioned a light at point *C* so that the edges of the light beam meet up exactly with the backdrop at segment *AB*.



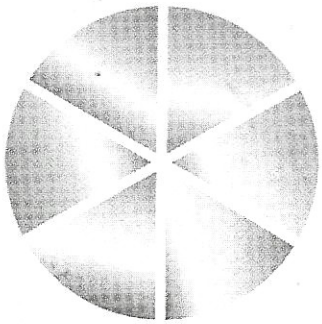
Noah wants to try different positions for the light to highlight different aspects of the sculpture, but he still wants the edges of the beam to exactly meet the endpoints of the backdrop. Find at least 3 other places Noah can place the light. Explain or show your reasoning.

Summary

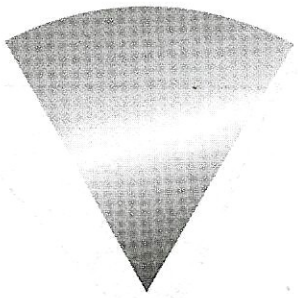
Putting It All Together

We can use sector areas to compare the value of product offers. Suppose the manager of a store wants to buy several dozen mirrors shaped like sectors of a circle to decorate the store. The manager is choosing between these 2 brands of mirrors at a tradeshow. Brand A's mirror has radius 14 inches and costs $35 for each sector. Brand B's mirror has radius 15 inches and costs $42. Which is the better deal in terms of cost per square inch of mirror?

brand A



brand B



Brand A's mirror is made from a circle cut into 6 congruent slices. Using the expression πr^2 with radius 14 inches, we find that the area of the full circle is 196π or about 616 square inches. Divide that by 6 to find that each sector-shaped mirror has an area of about 103 square inches. At $35 per sector, this mirror costs about $0.34 per square inch.

For brand B, we don't know how many slices the mirror was cut into. However, we can estimate the measure of the central angle using arc length and radius. Suppose the manager uses a flexible measuring tape to find that the length of the arc around the outside of the sector is 19 inches. The ratio of the arc length to the radius gives the measure of the central angle in radians, $\frac{19}{15}$. This is about 1.27 radians, but to avoid rounding errors, let's use the exact value, $\frac{19}{15}$, in our calculations.

Next, we can find the area of the sector with the formula $\frac{1}{2}r^2\theta$ where r is the radius and θ is the radian measure of the central angle. Substitute in our values to get $\frac{1}{2}(15)^2 \cdot \frac{19}{15}$, or 142.5 square centimeters. At $42 per sector, this mirror costs about $0.29 per square inch. Brand B's mirror is a better deal.

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Practice

Putting It All Together

1. Jada is finding the area of a sector with an angle $\frac{\pi}{4}$ radians and radius 8 units. She found the area of the whole circle, then found the fraction represented by the sector by dividing $\frac{\pi}{4}$ by 360. She multiplied this fraction by the total circle area. (Lesson 7-13)

   a. Do you agree with Jada's strategy? Explain your reasoning.

   b. Find the area of the sector.

2. Which of these pizza slices gives the best value (the most pizza per dollar spent)? (Lesson 7-13)

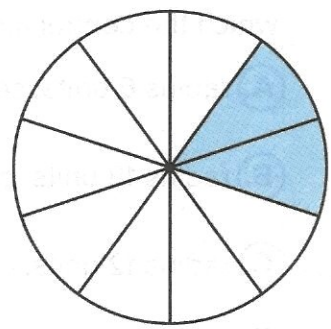
   (A.) a slice with a radius of 12 inches, central angle of 30°, and a cost of $3 per slice

   (B.) a slice with a radius of 8 inches, central angle of 45°, and a cost of $2 per slice

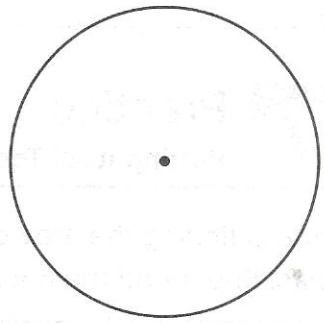
   (C.) a slice with a radius of 6 inches, central angle of $\frac{\pi}{3}$ radians, and a cost of $2 per slice

   (D.) a slice with a radius of 6 inches, central angle of $\frac{\pi}{4}$ radians, and a cost of $1 per slice

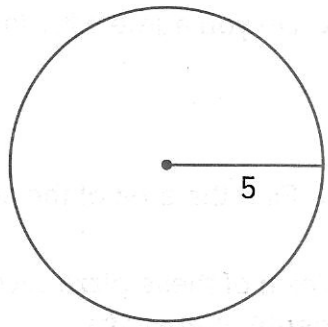
3. The circle in the image has been divided into congruent sectors. What is the measure of the central angle of the shaded region in radians? (Lesson 7-12)



4. In the circle, sketch a central angle that measures $\frac{5\pi}{3}$ radians. (Lesson 7-12)



5. The image shows a circle with radius 5 units. (Lesson 7-11)
   a. Draw a 180 degree central angle (a diameter) in the circle. What is the length of the arc defined by this angle?
   b. Use the arc length and the radius to calculate the radian measure of 180 degrees.
   c. Calculate the radian measure of a 360 degree angle. Explain or show your reasoning.



6. Complete the table. Each row represents a circle with a defined sector. (Lesson 7-9)

Sector Area	Radius	Central Angle
5π cm^2	5 cm	
12π cm^2		270 degrees
	12 cm	15 degrees

7. Several circles with central angles are described. Select all the circles for which the central angle defines arcs that have length 6π units. (Lesson 7-8)

   (A.) radius 6 units, central angle 180 degrees

   (B.) radius 18 units, central angle 60 degrees

   (C.) radius 12 units, central angle 90 degrees

   (D.) radius 3 units, central angle 120 degrees

   (E.) radius 4 units, central angle 270 degrees

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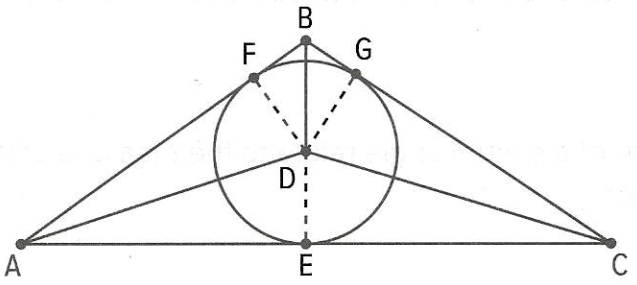
8. Triangle *ABC* is shown with its incenter at *D*. The inscribed circle's radius measures 2 units. The length of *AB* is 9 units. The length of *BC* is 10 units. The length of *AC* is 17 units. (Lesson 7-7)



a. What is the area of triangle *ABD*?

b. What is the area of triangle *BCD*?

9. Noah makes 3 statements about the incenter of a triangle. (Lesson 7-6)

For each statement, decide whether you agree with Noah. Explain your reasoning.

a. To find the incenter of a triangle, you must construct all 3 angle bisectors.

b. The incenter is always equidistant from the vertices of the triangle.

c. The incenter is always equidistant from each side of the triangle.

10. Elena is writing notes about central angles in circles. Help her finish her notes by answering the questions. **(Lesson 7-1)**

a. Where is the vertex of a central angle located in relation to the circle?

b. What line segments related to circles are contained in the rays that form a central angle?

c. How does the measure of a central angle relate to the measure of the arc it is associated with?

Learning Targets

Lesson	Learning Target(s)
7-1 Lines, Angles, and Curves	• I know what chords, arcs, and central angles are.
7-2 Inscribed Angles	• I can use the relationship between central and inscribed angles to calculate angle measures and prove geometric theorems. • I know that an inscribed angle is half the measure of the central angle that defines the same arc.
7-3 Tangent Lines	• I can use the relationship between tangent lines and radii to calculate angle measures and prove geometric theorems. • I know that a line tangent to a circle is perpendicular to the radius drawn to the point of tangency.
7-4 Quadrilaterals in Circles	• I can prove a theorem about opposite angles in quadrilaterals inscribed in circles.

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Lesson	Learning Target(s)
7-5 Triangles in Circles	• I can construct the circumscribed circle of a triangle. • I can explain why the perpendicular bisectors of a triangle's sides meet at a single point.
7-6 A Special Point	• I can explain why the angle bisectors of a triangle meet at a single point. • I know any point on an angle bisector is equidistant from the rays that form the angle.
7-7 Circles in Triangles	• I can construct the inscribed circle of a triangle.
7-8 Arcs and Sectors	• I can calculate lengths of arcs and areas of sectors in circles.
7-9 Part to Whole	• I can gather information about a sector to draw conclusions about the entire circle.

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Lesson	Learning Target(s)
7-10 Angles, Arcs, and Radii	• I know that when a circle is dilated, some ratios, like the ratio of the circumference to the diameter, stay constant.
7-11 A New Way to Measure Angles	• I know that the radian measure of an angle whose vertex is the center of a circle is the ratio of the length of the arc defined by the angle to the circle's radius.
7-12 Radian Sense	• I understand the relative sizes of angles measured in radians.
7-13 Using Radians	• I can calculate the area of a sector whose central angle measure is given in radians. • I know that the radian measure of an angle can be thought of as the slope of the line $\ell = \theta \cdot r$.
7-14 Putting It All Together	• I can use properties of circles to solve geometric problems.

Notes:

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